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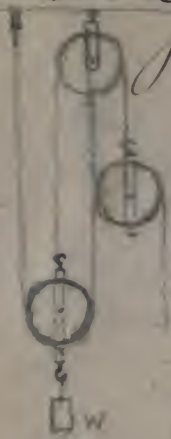

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AN

ELEMENTARY TREATISE

ON

MECHANICS;

DESIGNED AS A TEXT-BOOK FOR THE UNIVERSITY EXAMINATIONS FOR THE ORDINARY DEGREE OF B.A.

C2 Pts. in 1 vol. 3

PART I.

STATICS.

173731

BY

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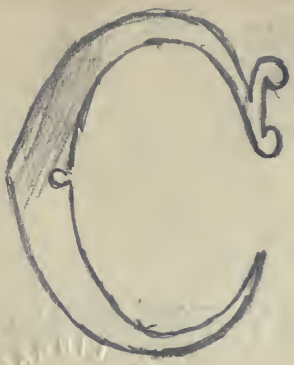

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I A material particle not acted upon by any force if at rest will remain so and if in motion it will move in a straight line with uniform velocity

II When uniform forces act continuously for a given time on material particles each produces in its own direction a change of momentum proportional to itself in magnitude

PREFACE TO FIRST EDITION.

This Treatise contains the text of the Lectures which I have delivered for some years past in University College.

As my design has been only to furnish to Students a text-book for such parts of the subject as are required for the ordinary degree of B.A. in Universities, I have not thought it advisable to burden this work with mere explanation or illustration, or to add examples; presuming that such, where necessary, will be furnished in the Lecture-room or by the Tutor.

University College,
Toronto,
April 1, 1858.

III If one material particle acts on another by some force the second reacts on the first in a direction equal to the same number and opposite in direction to that in which the first acts on it

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Principles that enter into the [illegible] of forces

CHAPTER I.

DEFINITIONS AND PRINCIPLES.

1. A *material particle* is a portion of matter occupying an indefinitely small space; or a geometrical point endowed with the properties of matter. Material particle.

All bodies may be geometrically conceived as made up of particles.

2. When the distance between two particles remains unchanged during any period of time, they are relatively at rest, and we conceive that they will continue so unless one or both be acted on by some cause to which we give the name of *force*. Force.

The state of rest or motion of a particle can only be conceived of in relation to others, but it is convenient to speak of it *absolutely* as being at rest or in motion, reference being understood to ourselves (or some particles in a known relation to ourselves), and changes of rest or motion are to be considered as produced by forces acting on the particle alone.

3. When a particle at rest is set in motion by a force, it will begin to move in a particular line, which we may define to be the *direction* or *line of action* of the force. The motion might be just prevented and the particle kept at rest, by a suitable force applied in an opposite direction. In this case the two forces are said to balance or counterbalance each other; and the magnitudes of two forces are said to be equal when each would separately counterbalance the same force. Direction and magnitude of a force. Equal forces.

4. Generally, when forces acting on any system of particles keep them at rest, the forces are said to counterbalance, or Statics, general problem of.

ence of Statics = Conditions of equilibrium or investigation of relations between forces and particles

to be in equilibrium; and the investigation of the relations among them in such case, or the conditions of equilibrium, constitutes the science of *Statics*.

Force, how measured.

5. Some convenient force being assumed as a standard or *unit*, the magnitude of any force is measured numerically with reference to the unit by the number of such units (acting simultaneously at a point and opposite to the force) which it will counterbalance. Thus, if the force will counterbalance n unit forces, its magnitude is said to be n.

This supposes n to be a whole number, and we can always take a unit-force of such magnitude that it shall be so; then, when any other force is taken as the unit, the magnitude of our original force will be expressed by the ratio (whether a whole number or a fraction) which its magnitude bears to that of the force assumed as unit.

In general, the term *pressure* may be used for a force thus statically considered and measured.

Weight.

6. It is found that on all bodies on the earth a pressure is exerted downwards, in a vertical direction: that is, in a direction perpendicular to the surface of still water at the place; and this pressure (which, for any particular body, is called the *weight* of that body) is invariable at the same place for the same body, whatever form, size or position the body be made to take. Hence the *weight* of some particular body may conveniently be assumed as a unit to which other pressures may be referred for measurement.

Unit of force.

In the English system the weight of a certain piece of brass carefully preserved as a standard, is called one pound Troy, and all other weights are referred to this. If lost, it might be restored from the knowledge that this *pound* being divided into 5760 *grains*, "a cubic inch of distilled water, of the temperature of 62° Fahrenheit, when the barometer is at 30 inches, weighs 252·724 grains."

Rigidity.

7. When a system of particles, or a body, is such that the relative distances of the particles undergo no change by the action of the forces applied to them in any manner, the system is said to be *rigidly* connected, and the body is called a *rigid* body, relatively to the forces concerned.

8. From the definitions laid down, it will be observed that three elements enter into every force: (1), its point of application, or the particle on which it acts; (2), its direction; (3), its magnitude. When these are known, the force is fully determined.

Experimental law.

9. The following is the fundamental law, deduced from experiment, on which the science of Statics is based:

If two equal forces act respectively on two particles, which are rigidly connected, in the line joining them but in opposite directions, they will counterbalance.

Hence, either of these forces may be transferred to the other particle, preserving the same direction, without alteration of its statical effect; or:

"*A force may be supposed to act at* ANY *point in its own line of action, the new point of application being rigidly connected with the former one*," and in this latter form the law is frequently stated.

10. The following consequences may be noted:

When a pressure is communicated by means of a straight rigid rod in direction of its length, the pressure is wholly effective in this direction, and may be supposed to act at any point of the rod.

Tension of a string.

When an inextensible string is stretched straight by a force at each end, these two forces must be equal, and their magnitude is independent of the length of the string. Also at *every* point of the string there are acting along it, in opposite directions, two forces equal to the former, and either of these is called the *tension* of the string, which is thus uniform throughout its whole length. The same is true when the string (if it be perfectly flexible) is stretched over a *smooth* surface.

A *smooth* surface is one which can exert a pressure at any point of it, only in direction of the normal at that point.

Forces can be represented by straight lines.

11. Since the three elements which serve to determine a pressure are in their nature identical with those which determine a straight line—namely, magnitude, direction, and point through which it is to be drawn—it follows that a straight line may properly be taken as the representative of a pressure. When, however, a line AB is so taken, it will be understood that the pressure acts in the direction from A towards B; if written BA, then from B towards A. Frequently also, the words "represented by" will be omitted, and we shall use "the force AB" to indicate the force represented in magnitude and direction by the line AB, acting in a direction from A towards B.

12. We now proceed to state the two problems of Statics which alone will be here investigated.

(1). The conditions of equilibrium for any set of forces acting on the same particle.

(2). The conditions of equilibrium when forces act on a rigid system of particles which has a fixed axis round which it can turn freely, the forces acting perpendicularly to this axis.

CHAPTER II.

FORCES ACTING AT A POINT.

Definition of resultant.

13. When forces act simultaneously on a particle at rest, if the particle begin to move, its motion will commence in a definite direction, and might be just prevented by a single force of suitable magnitude applied in an opposite direction. This force would then counterbalance the original set of forces, and a force equal and opposite to it would produce the same statical effect as the first set of forces, and is therefore termed their *resultant*.

14. Hence, when any set of forces acting at a point keep it at rest, since any one of them may be considered as counterbalancing all the rest, a force equal and opposite to any one of them is the resultant of all the others.

Condition of equilibrium.

15. Hence also the condition, in order that forces acting at a point may keep it at rest, is that the magnitude of their resultant shall be zero, or that their resultant shall vanish.

Forces in the same line.

16. When forces act in the same line and direction on a point, their resultant acts in the same direction, and its magnitude is equal to the sum of their magnitudes. If some of the forces be acting in the opposite direction, the magnitude of their resultant will be the difference between the sums of the magnitudes of those acting in the one direction and in the other, and it will act in the same direction as those forces whose sum is the greater. We can, however, indicate oppositeness of direction by attaching to the magnitudes of the forces the algebraic signs + and −; so that, any one force being considered *positive*, all forces in

that direction will also be considered *positive*, but forces in the opposite direction will be considered *negative*.

The above results may then be combined into the following:

Their resultant and

The resultant of any set of forces acting on a point in the same line, is the algebraic sum of the forces.

Condition of equilibrium.

17. Hence also the condition that the point may be kept at rest will be that

The algebraic sum of the forces shall be zero.

Equal oblique forces.

18. If two equal forces act in different directions at a point, their resultant will act in a direction bisecting the angle between their directions.

Any two forces.

19. *The principle of the* PARALLELOGRAM OF FORCES.

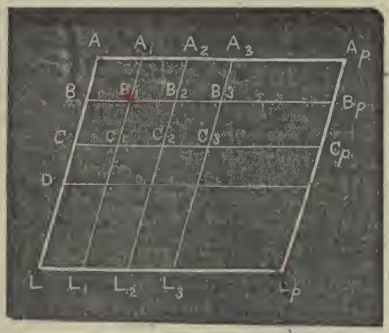
Parallelogram of forces.

If two forces acting on a particle be represented in magnitude and direction by two straight lines drawn from a point, and the parallelogram, of which these lines are adjacent sides, be completed, that diagonal which passes through the point will represent in magnitude and direction the resultant of the two forces.

Newton.

Direction of resultant.

Let the lines AA_p, AL be drawn representing in magnitude and direction two forces acting at A; and let p, q be the numbers denoting the magnitudes of the forces.



Duhamel's proof.

Divide AA_p into p equal parts in the points A_1, A_2, A_3, ..., and AL into q equal parts in the points B, C, D, ... : then each of these equal parts will represent in magnitude the unit force. Through these points, draw lines

parallel to the original lines, completing the parallelogram; and suppose all the lettered points of the figure rigidly connected.

Then, since the two forces represented by AA_1, AB, acting at A, are equal, the direction of their resultant bisects the angle between them, and it therefore acts in AB_1: it may then be supposed to act at B_1 (§ 9), and may there be again resolved into its original components, which will be represented by BB_1 and A_1B_1, of which the former may act at B, and the latter at A_1.

Proceeding in the same way with this latter force, A_1B_1, at A_1, and the force A_1A_2, which we may also take to act at A_1, we can replace these by B_1B_2 at B_1, and A_2B_2 at A_2.

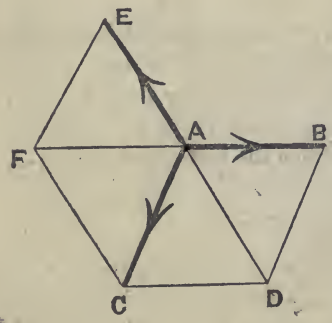
Proceeding in this manner we arrive at last at B_p, where we find the force A_pB_p and the set of forces BB_1, B_1B_2, B_2B_3..... (which latter make up the original force p) as the equivalents of p and AB at A.

Now taking up the set of forces in BB_p, and the force represented by BC at B, we transform them by the same process into B_pC_p and the set in CC_p at C_p.

Following this method, we arrive at last at L_p, where we have for the equivalents of the original forces the sets of forces in LL_p and A_pL_p, which may be supposed all to act at L_p and their magnitudes are p and q. Hence we have transformed the original forces p and q acting at A to the same forces acting at L_p in parallel directions to the former, and this without alteration of their statical effect. Hence L_p must be a point in the direction of the resultant of the original forces at A; that is, AL_p is the direction of the resultant, which proves the principle enunciated, so far as the direction of the resultant is concerned.

Magnitude of resultant.

Let AB, AC, represent the two forces acting at A. Complete the parallelogram $AC\,DB$. Then AD is the direction in which the resultant acts, and we have now to prove that AD represents also its magnitude.



In DA produced backwards take AE to represent this magnitude, so that a force represented by AE will be equal and opposite to the resultant; and the three forces represented by AB, AC, AE, will keep the point A at rest, and each one of them is equal and opposite to the resultant of the other two.

Complete the parallelogram $AEFC$; then, AF is the direction of the resultant of the forces AE, AC, and is therefore opposite to AB. Hence, FAB is a straight line, and therefore $FACD$ a parallelogram. Hence, the lines AE and AD are equal, being each equal to FC: but AE was taken to represent in magnitude the resultant of AB, AC, and consequently AD also represents it in magnitude. *Q.E.D.*

Resolution of a force.

20. Conversely, a force acting at a point can be resolved into an equivalent pair of forces at that point in an infinite number of ways; for, taking a line drawn from a point to represent the force, and constructing on it as diagonal *any* parallelogram, the two adjacent sides terminating at this point will represent an equivalent pair of forces.

If this pair consist of two forces acting in perpendicular directions, each of them is called the *effective* part of the original force in this direction.

Thus, if R be the force at A, represented by AD, and it be resolved into two forces in perpendicular directions—namely, X along AB and Y along AC; then X and Y are the effective parts of R resolved along AB and AC respectively. Completing the rectangle $ACDB$, X and Y will be respectively represented by AB, AC, and, calling the angle BAD, θ, we have from the right-angled triangle BAD,

$$X = R \cos \theta, \quad Y = R \sin \theta.$$

Rule for.

21. Hence, to find the effective part of a force in any given direction, or, more briefly, to resolve a force in any given direction, *multiply its magnitude by the cosine of the angle contained between its direction and the given direction*; and to resolve a force perpendicularly to a given direction, *multiply it by the sine of this angle*.

22. So also from the same figure we obtain the resultant (R) of two perpendicular forces (X, Y), and the angle (θ) which its direction makes with one of them (X); for

$$R^2 = X^2 + Y^2; \text{ and, } \tan \theta = \frac{Y}{X}.$$

23. When any number of forces act at a point, their whole effect in any direction will be the algebraic sum of the separate resolved forces in this direction, which will evidently therefore be equal to their resultant resolved in the same direction.

Hence also the algebraic sum of the separate resolved forces in direction of the resultant is the resultant itself, and the corresponding sum in a direction perpendicular to the resultant is zero.

Resultant of any forces in one plane, and

24. *To find the magnitude and direction of the resultant of any forces acting at a point, their directions being all in one plane.*

Taking any two directions at right angles to each other, let each force be resolved into its components in these directions. Let the algebraic sum of these resolved parts in the one direction be X, and in the other Y.

Then the whole set of forces is equivalent to the two X, Y.

Hence, if R be the resultant of the whole set, and therefore also of the two X, Y, and θ the angle it makes with the direction of X, the equations in § 22 give

$$R^2 = X^2 + Y^2, \tan\theta = \frac{Y}{X},$$

which determine the resultant in magnitude and direction. We have also the equivalent relations

$$X = R\cos\theta, \quad Y = R\sin\theta.$$

Conditions of equilibrium.

25. *To find the conditions of equilibrium when any forces act at a point, their directions being in one plane.*

Retaining the notation and method of the last article, since the only condition, in order that the point acted on by the forces may be kept at rest, is (§ 15) that the resultant of the forces must be zero; that is, $R = 0$, we have

$$X = 0, \quad Y = 0.$$

And, conversely, if $X = 0$, and $Y = 0$, then we also have $R = 0$, and the point will be kept at rest; hence the necessary and sufficient conditions of equilibrium are that

The algebraic sums of the forces resolved into two perpendicular directions shall separately vanish.

This principle will be cited under the name of "The vanishing of the resultant."

26. The process can be readily extended to forces not all acting in one plane.

Thus, if three forces not in one plane act at a point, and three lines be drawn representing them in magnitude and direction; then, if the parallelopiped, of which these lines are adjacent edges, be

completed, that diagonal of it which passes through the point will represent in magnitude and direction the resultant of the forces.

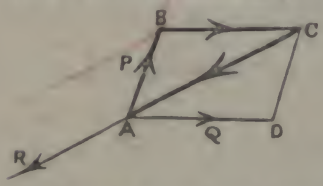
Also, if any number of forces act at a point, the necessary and sufficient conditions of equilibrium are that the algebraic sums of the Forces resolved along three mutually perpendicular directions shall separately vanish.

The following propositions are historically interesting, though included in what has preceded.

27. *Triangle of forces.* Triangle of forces. Stevinus

If the directions of three forces acting at a point be parallel to the sides of a triangle taken in order, and their magnitudes be proportional to these sides, they will keep the point at rest.

For if ABC be the triangle, and A the point at which the forces act; then, completing the parallelogram $ABCD$, the two forces represented by AB, BC, will be represented by AB, AD, and their resultant by AC, which is equal and opposite to CA, the third force.



28. Conversely. If three forces acting at a point and keeping it at rest, be represented in direction by the sides of a triangle taken in order, these sides will represent them also in magnitude.

29. Hence, all problems relative to three forces keeping a point at rest are reduced to the solution of a plane triangle. Thus, if P, Q, R, be the forces, and the angle between P and Q be represented by (P, Q); then the angles of the triangle in the above proposition are the supplements of the angles between the forces; and, since the sine of an angle is equal to that of its supplement, and the cosine of an angle is the cosine of the supplement with opposite sign, we have (Trig. § 34, 40.)

$$\frac{P}{\text{Sin }(Q,R)} = \frac{Q}{\text{Sin }(R,P)} = \frac{R}{\text{Sin }(P,Q)},$$ Lamí's formulas.

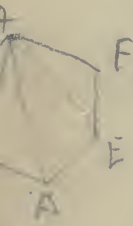
and also the equivalent expressions—

$$R^2 = P^2 + Q^2 + 2\ P\ Q \cos\ (P,\ Q)$$
$$P^2 = Q^2 + R^2 + 2\ Q\ R \cos\ (Q,\ R)$$
$$Q^2 = R^2 + P^2 + 2\ R\ P \cos\ (R,\ P)$$

Polygon of forces.

30. Polygon of forces.

If forces acting on a point be represented in magnitude and direction by the sides of a polygon, taken in order, they will keep the point at rest.

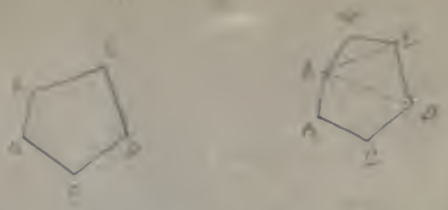


For if $ABCDEF$ be the polygon, the forces AB, BC, have for their resultant AC; and the resultant of this and CD is AD; and so on till we come to the last side, which is equal and opposite to the resultant of all the previous ones.

Hence the proposition, as well as its converse, is established.

31. In this way, the resultant of any number of forces at a point can be constructed geometrically; for, having drawn consecutive lines, so that, taken in order, they are parallel to, in the same direction with, and proportional in magnitude to the forces; the line drawn to complete the polygon will represent in magnitude and in reversed direction the resultant required.

It may be noticed that the polygon referred to need not be a *plane* one, neither are re-entering angles or crossed sides excluded, nor does a change of the order in which the forces are taken affect the result.



P. OB = Q OC

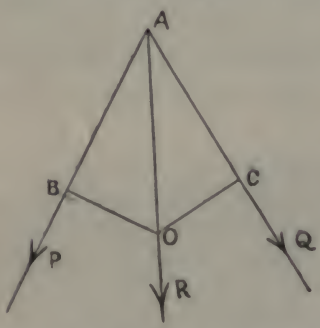
CHAPTER III.

FORCES IN ONE PLANE ACTING ON A SYSTEM OF RIGIDLY CONNECTED POINTS, WHICH CAN TURN FREELY ABOUT A FIXED POINT IN THE PLANE.

Condition of equilibrium when the two forces meet.

32. *Two intersecting forces act on a rigid system, in the same plane with a fixed point round which the system can turn.*

Let O be the fixed point; P, Q, two forces in the same plane with O, their directions intersecting in A, at which point, rigidly connected with O, they may be supposed to act.



Then if R be the resultant of P, Q, in order that the point A and the whole system with which it is rigidly connected may be kept at rest, it is necessary and sufficient that the direction of R shall pass through the fixed point O: that is, AO must be the direction of R. Draw OB, OC perpendicular to the directions of P, Q. Then, resolving the forces at A in a direction perpendicular to AO, we have (§ 21, 23):

$$P \sin OAB - Q \sin OAC = 0, \text{ and therefore}$$

$$P.\ OB - Q.\ OC = 0, \text{ or,}$$

$$P.\ OB = Q.\ OC.$$

Moment defined.

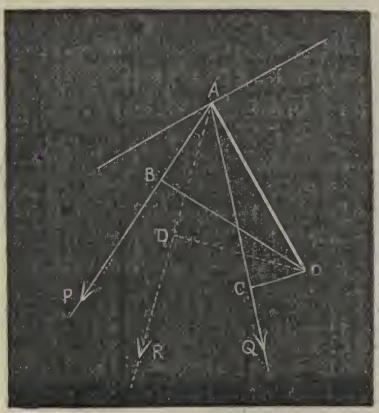
33. The product $P.\ OB$, which is the product of the number expressing the magnitude of the force, and the length of the perpendicular dropped from the point O upon its direction, is called *the moment of the force about that point.*

Hence the above result may be expressed by saying that when the two forces keep the system at rest, *their moments about the fixed point are equal*, the forces tending to turn the system in opposite directions about the point: but if we indicate this oppositeness of direction by difference of algebraic sign, so that the moment of one force which tends to turn the system round in *one* direction being considered *positive*, that of another force tending to turn it in the opposite direction will be considered *negative;* we may still more briefly express it in the form:

The algebraic sum of the moments of the two forces round the point must be zero.

Moment of resultant of two forces which intersect is equal to sum of moments of forces.

34. *The moment of the resultant of two intersecting forces round any point in their plane is equal to the algebraic sum of the moments of the forces.*



Let P, Q be the two forces intersecting in A; O, any point in their plane; R, their resultant.

Draw the perpendiculars OB, OC, OD.

Then (§ 23) the resultant resolved in any direction being equal to the algebraic sum of the resolved forces in that direction, let them be resolved perpendicularly to OA. Hence, from § 21,

$$R \sin OAD = P \sin OAB + Q \sin OAC;$$

and therefore

$$R.\,OD = P.\,OB + Q.\,OC,$$

which proves the proposition.

Cor. We have taken the case in which the moments of the forces have the same sign, the proof in this case being sufficient for all. Where, as in the figure, the two forces tend to turn the system in opposite directions, their moments will bear different signs, and we have, by the same process as above,

$R.\,OD = P.\,OB - Q.\,OC$;

and the direction in which the system tends to turn will be indicated by the sign of the moment $R.\,OD$ found from this expression.

35. *Two forces act in parallel directions on a rigid system.* Two parallel forces are equivalent to a single force

Let P, Q be the two forces; O, any point in their plane. Draw OCB perpendicular to the forces. At B, C, apply two equal and opposite forces T; these will in no way affect the system.

Let R be the resultant of P, T, acting at B; and S that of Q, T, at C. Then the directions of R, S will in general meet: let them do so in A, and suppose them to act at this point. They can now be here resolved into their original com-

parallel to them,

whose magnitude is their sum

ponents, P, T'; Q, T; of which the forces T, T', being equal and opposite, may be removed altogether, leaving the forces P, Q acting in a direction parallel to their original direction, and combining into a single force $(P + Q)$.

Again, the moment of this single force $(P + Q)$ about O is equal to the algebraic sum of those of its components R and S (§ 34); and the moment of R is equal to the sum of the moments of its components, namely, P and T'; and so is that of S to the sum of the moments of Q and T; among which moments those of the forces T, T' destroy each other, leaving the algebraic sum of the moments of the original forces P, Q equal to the moment of the single force $(P + Q)$, which has been shown to be their equivalent.

and whose moment is the sum of their moments.

If the forces P, Q had been taken acting in opposite directions, we should have found by the same process that the single equivalent force had for its magnitude the difference of those of P and Q, and acted in the direction of the greater force, but that its moment was still equal to the algebraic sum of their moments.

If, therefore, we now extend to parallel forces the same method of indicating oppositeness of direction by difference of sign, which was used in the case of forces acting in the same line, we can include the above cases in a single statement, as follows:

Two parallel forces acting on a rigid system are equivalent to a single parallel force which is equal to their algebraic sum, and whose moment round any point in the plane of the forces is equal to the algebraic sum of the moments of the two forces.

Exception A couple.

In one case, however, the above process becomes nugatory, which is when the two forces are parallel, equal, and opposite. Such a pair of forces is called a *couple*, and the case must be excluded from our general statement.

Any two forces, resultant of.

36 If to this single equivalent force in § 35, we give the name of *resultant*, we can now include the results of the two last articles in one statement:

Any two forces in the same plane acting on a rigid system (unless they form a couple) are equivalent to a single resultant force, whose moment round any point in their plane is equal to the algebraic sum of their moments round this point.

37. *Any forces act in one plane on a rigid system.* Any forces in one plane

Taking any two of these, we find their resultant, its moment being the sum of the moments of the two round any assumed point in the plane; combining this resultant with a third to form a new resultant, whose moment will be the sum of those of the three forces; and this again with a fourth; and so on till we have taken all the forces, we are left at last with a single resultant only, whose moment is equal to the sum of the moments of all the forces. In thus proceeding, we must avoid combining with any one of the partial resultants a force which would form with it a couple; and this we can always do by taking instead of this force another one which will *not* form a couple, for if it did, there would then be *two* equal and parallel forces, *not* opposite, and these two could be combined into one which would now no longer form a couple with the resultant spoken of; we can thus always evade forming a couple until we have combined all the forces but one, and it may happen that this one is equal, parallel, and opposite to the resultant we have obtained from all the rest, so that we have a *couple* remaining.

Hence, *any set of forces acting in one plane on a rigid system are either reducible to a couple, or else to a single resultant force, whose moment round any point in the plane is equal to the algebraic sum of the moments of the forces round that point.* are reducible to single resultant, in general,

38. *To find the conditions of equilibrium when forces in one plane act on a rigid system which can turn freely about a fixed point in that plane.*

The forces are reducible either to a couple or to a single resultant force. and for equilibrium.

In the former case, equilibrium is not possible; in the latter, equilibrium will subsist if the resultant either be zero or pass through the fixed point, and each of these suppositions will make its moment about this point vanish, and therefore also the algebraic sum of the moments of all the forces, to which sum it has been shown to be equal.

Hence the necessary and sufficient condition of equilibrium is, that

The sum of their moments vanishes.

The algebraic sum of the moments of all the forces about the fixed point must vanish.

This principle will always be quoted by the name of "the vanishing of moments."

39. The same principle may easily be seen to apply when the rigid body is capable of turning about a fixed straight line or axis, and the forces are not all in one plane but are perpendicular to this axis. The moment of each force being taken about that point of the axis which is cut by a perpendicular plane containing the force, we can state the condition of equilibrium in the form:

The algebraic sum of the moments of the forces about the fixed axis must vanish.

[This completes the solution of the two problems which it was proposed (§ 12) to investigate, but the solution of the more general problem of statics may here be noted.

When a rigid body can turn freely about a fixed point in itself, two moments about different axes passing through the point are equivalent to a single moment about another axis, this resultant moment being determined by the same method as in the *parallelogram of forces*, viz.:—by drawing lines from a point in directions of the two axes and proportional in length to the magnitudes of the moments, and completing the parallelogram of which these are adjacent sides; then the diagonal represents the direction of the resultant axis and the magnitude of the moment about it. Hence any set of forces acting on such a body are equivalent to a single resultant moment, and the condition that they may keep it at rest is that this resultant moment shall vanish, and (as in § 26) the necessary and sufficient conditions that this shall be the case are that *the algebraic sums of the moments about three mutually perpendicular axes must separately vanish.*

To recapitulate the conditions of equilibrium :

(1.) When forces act at a point and keep it at rest, the resultant force must vanish, and therefore *the algebraic sums of the forces resolved along three mutually perpendicular lines must separately vanish.*

(2.) When forces act on a rigid body which can turn freely about a fixed point, the resultant moment must vanish, and therefore *the algebraic sums of the moments of the forces about three mutually perpendicular axes must separately vanish.*

(3.) When forces act on a rigid body which is perfectly free, and keep it at rest, the preceding conditions must both be fulfilled ; that is, the forces must be such that (1) if they acted at a point retaining their directions and magnitudes, they would keep it at rest ; and (2) if a point of the body were fixed and the body free to turn about it, they would keep it at rest. Thus involving on the whole six necessary and sufficient conditions for equilibrium.

(4). When any system of rigid bodies, acting on each other by contact or in other ways, is in equilibrium under impressed forces, the principle laid down in Newton's *Third Law of Motion* (hereafter quoted) asserts that the actions of these bodies are mutual, equal, and opposite ; and therefore by forming the conditions of equilibrium for each body separately we obtain enough equations to eliminate their mutual forces and leave the resulting conditions among the impressed forces.

Although this is the complete solution of all the statical problems which arise in the consideration of rigid systems, it must be noted that in the practical application to the cases which occur in nature, these results can only be regarded as a first approximation, because no such thing as a rigid body is known to exist. All natural bodies are more or less yielding, and this circumstance gives rise to problems of extreme difficulty and complexity, in the solution of which not much progress has yet been made.]

CHAPTER IV.

CENTRE OF PARALLEL FORCES, AND OF GRAVITY.

Resultant of two parallel forces has its moment about a perpendicular line

40. It has been shown that two parallel forces (not forming a couple) acting on a rigid system, have for resultant a single force in the same plane; its direction being parallel to that of the two, its magnitude being the algebraic sum of their magnitudes, and its moment about any point in this plane being the algebraic sum of their moments about this point.

Also, the moment of a force about a line to which its direction is perpendicular has been defined to be the product of the number expressing the magnitude of the force, by the perpendicular distance between its direction and the line.

41. It will now be shown that the moment of this resultant of two parallel forces, *about any line* perpendicular to their direction, is equal to the algebraic sum of the moments of the two forces about this line.

equal to the sum of the

Supposing the forces P, Q, to be acting in the same direction perpendicularly to the plane of the figure and meeting this plane in the points B, C; and their resultant R (which $= P + Q$, and acts parallel to and in the same plane with them), to meet the plane of the figure in A. Let bac be any line in this plane, and draw to it the perpendiculars Aa, Bb, Cc. Then, if BAC be parallel to bac, Aa, Bb, Cc, are all equal, and the proposition is manifestly true, for

$$R.\ Aa = (P + Q)\ Aa = P.\ Bb + Q.\ Cc.$$

But if BAC be not parallel to bac, let them meet in O. Then, O being a point in the plane of the forces, the moment of R round O is equal to the sum of those of P and Q round it, and therefore

$$R.\ OA = P.\ OB + Q.\ OC.$$

But by similar triangles

$$\frac{Aa}{OA} = \frac{Bb}{OB} = \frac{Cc}{OC}$$

and therefore

$$R.\ Aa = P.\ Bb + Q.\ Cc,$$

moments of the forces.

which proves the proposition for this case, and the proof holds good also for the cases where the forces or their moments are in opposite directions, having due regard to algebraic sign.

42. *Any parallel forces, acting on a rigid system, are either reducible to a couple or else to a single resultant force which acts in a parallel direction, its magnitude being the algebraic sum of the magnitudes of all the forces, and its moment about any assumed line perpendicular to their direction, being equal to the algebraic sum of the moments of all the forces about this line.*

Any number of parallel forces have in general a single resultant.

Taking any two of the forces (which do not form a couple), we find their resultant, which acts in a parallel direction, its magnitude being the algebraic sum of their magnitudes, and its moment, about any assumed line perpendicular to the direction of the forces, being equal to the algebraic sum of their moments about this line: combining this resultant with any third force to form a new resultant, and this again with a fourth, and so on as in § 37, we arrive at last either at a couple or a single resultant force acting in a parallel direction, its magnitude being equal to the algebraic sum of the magnitudes of all the forces, and its moment about the assumed line being equal to the algebraic sum of all their moments about this line.

Centre of parallel forces.

43. *The centre of parallel forces.*

When given parallel forces, acting at given points of a rigidly connected system, are reducible to a single resultant, its direction passes through a point whose position is invariable with regard to the points of the system, whatever be the direction of the forces.

Take any two of the forces P, Q, acting at the given points B, C. Join BC and let their resultant R cut it in A. Then, the moment of R about any point in the plane being equal to the algebraic sum of the moments of P, Q, about this point; let these moments be taken about A.

The moment of R about A is zero; hence, drawing bAc perpendicular to the direction of the forces,

$$P.\,Ab - Q.\,Ac = 0,$$

and, by similar triangles,

$$\frac{AB}{AC} = \frac{Ab}{Ac}, \text{ and therefore,}$$

$$= \frac{Q}{P}.$$

Hence BC, which is given, is cut in the point A in a ratio which is independent of the direction of the forces with regard to B C, and the position of A is therefore *given* with regard to B and C.

Now, taking any third force, acting at D, we may combine it with the resultant of P and Q, and the point in which the new resultant cuts AD will be given in position with regard to A and D or to A, B and C.

And thus we may go on till we arrive at the final resultant.

Hence, the proposition as enunciated is true.

This point is called the *centre of parallel forces.*

44. If this point—the centre of parallel forces—in a given system be rigidly connected with the system and supported or fixed, the system will be kept at rest, and will remain so when the forces are turned about their points of action into any other direction. It will also still be at rest if it be turned about this point into any other position, the forces acting always at the same points of the system and being always parallel to each other, though their directions may be varied at pleasure. The system may be turned about it.

The pressure supported by this fixed centre is evidently the algebraic sum of the forces, and the algebraic sum of their moments about any line through this point vanishes.

45. *The centre of gravity.* Centre of gravity.

When the only forces acting on a system are the weights of the several particles of that system, if we suppose the vertically-downward directions in which these weights act to be parallel to each other, and the weight of any particle to be independent of its position; then, since the forces all act in the same direction, they have a single resultant which is equal to their sum, that is, to the weight of the whole system, and acts vertically downwards through the *centre of parallel forces*, which is in this case called the *centre of gravity*.

[Neither of the suppositions above made is true.

The vertically-downward directions of the weights are not parallel, and the weight of a particle may change when its position is changed; but the errors thus introduced are insensible when the system of particles under consideration occupies a comparatively small space on the earth, and in the cases which we are about to discuss may be neglected.]

46. The statical effect, therefore, of any rigid system will not be altered by supposing it to be without weight, and the whole weight to be collected at its centre of gravity and there to act—this point, however, being rigidly connected with the system. Whole weight may be collected at:

We may also, without alteration of the statical effect, conceive the system to be geometrically divided into any number of systems, and the weight of each of these to be collected at

its own centre of gravity and there act, these partial centres of gravity being rigidly connected with each other and the system.

System balances about in all positions:

47. Also, if the centre of gravity of a system be supported or fixed, the system will balance about this point in all positions under the sole action of the weights of the parts of the system, these being rigidly connected with each other and the centre of gravity, and this is sometimes made the definition of the *centre of gravity.*

How found.

48. The position of the centre of gravity relative to a given system will be determined from the consideration, that, placing the system so that any given line in it shall be horizontal, and equating the moment of the whole weight collected at the centre of gravity with the moments of the several weights of the particles about this line, the distance of the centre of gravity from the vertical plane passing through this line will be found. Taking thus three planes in succession intersecting in a point, the distances of the centre of gravity from each of these planes can be found, and its position therefore determined.

49. Since the position of the centre of gravity in the system depends on the relative and not the absolute weights of its parts, this position will not be affected by increasing or diminishing proportionally these weights.

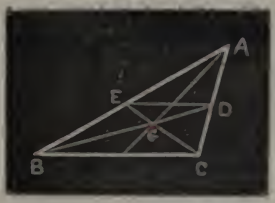
Of a uniform body.

50. If a rigid body be of uniform density: that is, if the weight of a given volume of its substance be the same in every part of the body; then, if there be a line about which the form of the body is symmetrical, the centre of gravity will be in that line; and if there be two such lines, the centre of gravity will be their intersection. Thus the centre of gravity of a circle or sphere is the centre; of a parallelogram or parallelopiped, the intersection of its diagonals; of a regular prism or cylinder, the middle point of its axis.

51. If a uniform body balance in every position about a line, the centre of gravity will lie in that line; and if about two such lines separately, it will be their intersection. Thus a triangular area will balance about a line drawn from one angle to bisect the opposite side, for the triangle can be generated by a line moving parallel to one side, and the small area gene-

rated at any stage of its motion will balance about the line which bisects it. Hence the centre of gravity of a triangular area is the intersection of lines drawn from the angles to bisect the opposite sides, and this intersection is at a distance from the angle of two-thirds of the bisecter drawn from it.

Of a triangular area.



For let ABC be the triangle, and BD, CE bisect AC, AB, and meet in G. Then G is the centre of gravity.

Join ED, which is parallel to BC (Eucl. B. VI. 2),

Then $\frac{BG}{GD} = \frac{BC}{ED}$, by similar triangles BGC, EGD

$= \frac{CA}{AD}$, by similar triangles ACB, ADE

$= \frac{2}{1}$.

Hence BG, being double of GD, is $\frac{2}{3}$ BD.

The same point G is also the centre of gravity of three equal bodies placed at the points A, B, C.

Cor. In this way can the centre of gravity of any polygonal area be found; for, dividing the figure into triangles, the weight of each of these may be supposed collected at its own centre of gravity, and the centre of gravity of the whole figure will be that of these weights, considered as heavy particles situated in those points.

Of any polygonal area.

The method of finding this latter will be treated in the following article.

52. *To find the centre of gravity of a system of particles all in one plane.*

Of any heavy particles in one plane.

Let Ox, Oy be two perpendicular lines in this plane, with regard to which the positions of the particles are known.

Let P be the place of one of the particles, w its weight.

Draw PN, PM perpendicular to Ox, Oy, respectively, and denote PM by x, PN by y.

Suppose the plane of the figure to be horizontal; then Ox is a line perpendicular to the direction of the weights, and therefore (§ 36) the moment about Ox of the whole weight collected at the centre of gravity is equal to the algebraic sum of the separate moments about it. Hence if W be the whole weight, and the distance of the centre of gravity from Ox be denoted by $\overline{y}$, we have

$$W.\overline{y} = \Sigma\,(w.y)$$

and

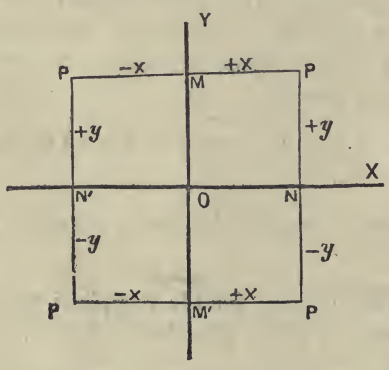
$$\overline{y} = \frac{\Sigma\,(w.y)}{W}$$

where Σ denotes the algebraic sum of all the products corresponding to that within the bracket. Also, if a moment be reckoned positive when P is above the line Ox, it will plainly be negative when P is below the line, and the difference in sign of the moments will therefore at once be indicated by considering a y positive when drawn upwards from Ox, and negative when downwards.

Similarly, by taking moments round Oy, if $\overline{x}$ be the distance of the centre of gravity from Oy, we have

$$\overline{x} = \frac{\Sigma\,(w.x)}{W}$$

where x will be considered positive when drawn to the right of Oy, negative when to the left.



The distances of the centre of gravity from these two lines being thus found, and the directions in which these distances are drawn being indicated by the signs with which they are affected, the position of the centre of gravity is fully determined.

Cor. If the particles all lie in the same line, take this for Ox. Then, every y being zero, $\overline{y}$ is so also, and the centre of gravity is in Ox, its distance from O being given by

Of particles in a straight line.

$$\overline{x} = \frac{\Sigma(w.\,x)}{W}$$

The following independent proof of this may be noted:

53. Let Ox be the line in which the particles lie, O being any point from which the distances of the particles are known, and let this line be placed horizontal. Let x be the distance from O of the particle whose weight is w.

Let W be the whole weight, and $\overline{x}$ the distance of the centre of gravity from O.

Draw another horizontal line from O perpendicular to Ox. This line will then be perpendicular to the direction of the weights, and the moment about it of the whole weight collected at the centre of gravity will be equal to the algebraic sum of the moments of the several weights. Hence we have

$$W.\overline{x} = \Sigma(w.\,x)$$

or

$$\overline{x} = \frac{\Sigma(w.\,x)}{W}$$

where Σ denotes the algebraic sum of all the products corresponding to that within the bracket. Also, if a moment be reckoned positive when the particle is on one side of O, that of a particle on the other side of O will be negative, and the difference in algebraic sign of the moments will therefore at once be indicated by considering the x's of the particles to be positive or negative according as they lie on one or the other side of O.

54. *When a rigid body rests suspended from or supported by a fixed point, and acted on only by its weight, the vertical line drawn through the centre of gravity will pass through the point of suspension or support; and, conversely.*

Heavy body suspended freely, its centre of gravity is vertically above or below the point of suspension.

For the weight of the body may be supposed collected at its centre of gravity, and there to act vertically downwards; and the necessary and sufficient condition of equilibrium is that its moment about the fixed point must vanish, which requires that its direction shall pass through this point.

Cor. 1. When the centre of gravity is vertically *under* the point of support or suspension, if the body be slightly disturbed from rest, the moment of the weight will tend to bring it back again to its original position. The equilibrium is therefore said to be *stable*. When the centre of gravity is vertically *above*, the contrary takes place, and the equilibrium is *unstable*.

Cor. 2. This affords a practical method of finding the centre of gravity of any plane area. Thus, suspending it freely from any one point, trace on it, when at rest, the direction of the vertical passing through this point: then, taking any other point (not in this line) for a new point of suspension, trace also the vertical through it. The intersection of the two lines thus drawn is the centre of the gravity required.

Body placed on a horizontal plane, will it stand or fall over.

55. *When a rigid body, having a plane base, is placed with this in contact with a fixed horizontal plane, and is acted on only by its own weight, it will stand or fall according as the vertical through its centre of gravity passes within or without the base.*

By the base is here meant the figure included by a string stretched completely round the outside of the plane section of the body which is in contact with the horizontal plane.

If the body fall over, it must begin to turn round some tangent to the curve formed by this string, and the moment of the weight, supposed collected at the centre of gravity, must tend about this tangent in a direction *from* the inside *towards* the outside of the area of the base, and the vertical through the centre of gravity will pass outside the base.

Also, when this vertical passes outside, the body must fall over; but if this vertical pass within the base, the moment of the weight about *every* tangent to the string tends in direction from the outside towards the inside, and the body cannot fall over.

CHAPTER V.

THE MECHANICAL POWERS.

56. The science of *Mechanics* was (as its name implies) in old times restricted to the employment of force on *machines*, and it is still usual to treat of the *simple machines*, or *mechanical powers* as they are sometimes called, under six classes, namely—the lever, the wheel and axle, the pulleys, the inclined plane, the screw, and the wedge. Of these, the wedge will not be here considered, as in its practical application the investigation on the principles of the foregoing chapters would be of small utility. The machine

When a power P sustains on any one of these machines a weight W, the ratio $W : P$ is called the *mechanical advantage* of the machine; and the machine is said to gain or lose advantage according as this ratio is greater or less than unity. Mechanical advantage defined.

In the following investigations, bodies will be supposed rigid, surfaces smooth, strings perfectly flexible and of insensible size, and the parts of the machine to be without weight, unless otherwise specified.

57. *The lever.*

A straight lever is a rod capable of turning freely in one plane about a point in itself which is fixed. This fixed point is called the *fulcrum*. Straight lever. Archimedes and DaVinci.

Case 1.—The weight W at one end of the lever supported by a weight P at the other end. Fig. 1.

BAC the lever; A the fulcrum.

Draw bAc horizontal, and therefore perpendicular to the direction of the weights.

Then by the vanishing of moments,

$$P.\,Ab - W.\,Ac = 0,$$

or $$P.\,Ab = W.\,Ac.$$

But, by similar triangles, $\frac{AB}{Ab} = \frac{AC}{Ac}$;

and therefore $$P.\,AB = W.\,AC.$$

Cor. 1. The pressure on the fulcrum is the weight $(P + W)$, acting vertically downwards.

Cor. 2. Since the relation $P.\,AB = W.\,AC$ does not involve the angle at which the lever is inclined to the horizon, it follows that if the lever be at rest in any one position (except a vertical one), on being turned into any other position it will still be at rest.

Fig. 2. CASE II.—The power P and the weight W acting in opposite (but parallel) directions, and the weight nearer to the fulcrum than the power.

Using the same construction and reasoning as in the former case, we have here also

$$P.\,AB = W.\,AC.$$

Cor. 1. The pressure on the fulcrum is here $W - P$, acting vertically downwards. The second corollary also holds.

Fig. 3. CASE III.—The power P and the weight W acting in opposite but parallel directions, and the power nearer to the fulcrum than the weight. As before, we have

$$P.\,AB = W.\,AC.$$

Cor. 1. The pressure on the fulcrum is $P - W$ and acts vertically upwards. The second corollary also holds.

Mech. adv. 58. In all these cases, the mechanical advantage $\left(\frac{W}{P}\right)$ is $\frac{AB}{AC}$ or the ratio of the arms of the power and weight. In

Case I. this ratio may be either equal to, greater, or less than unity; but in Case II. it is always greater, and in Case III. less: hence, advantage is always gained in the second case and lost in the third, but may be either gained or lost in the first.

59. If the weight of the lever (w) be taken into account, it may be supposed collected at the centre of gravity G w(hich will be the middle point if the lever be uniform). Lever supposed heavy.

Let the vertical through G meet the horizontal Abc in g.

Then, by the vanishing of moments about A,

$$P.\,Ab + w.\,Ag - W.\,Ac = 0;$$

but by similar triangles,

$$\frac{AB}{Ab} = \frac{AG}{Ag} = \frac{AC}{Ac}, \text{ and therefore,}$$

$$P.\,AB + w.\,AG - W.\,AC = 0,$$

or,
$$P.\,AB + w.\,AG = W.\,AC.$$

Similarly, in Cases II. and III. we should find

$$P.\,AB = W.\,AC + w.\,AG.$$

Here also the lever will balance in all positions about the fulcrum.

60. In the common balance, which consists of a heavy beam, having scale pans suspended at its ends, and balancing about a horizontal knife edge, the pans and arms of the beams are made perfectly equal and similar on each side of the edge, but the centre of gravity of the beam is made to fall vertically below the knife-edge when the beam is horizontal. The beam will therefore rest in a horizontal position only when the pans are loaded with equal weights; and if then disturbed from this position, the moment of its own weight brings it back, so that the equilibrium is *stable*. Common balance.

61. In the common or Roman steelyard, a heavy beam has attached to it a knife-edge which is supported as a fulcrum; a weight runs along the upper straight edge of the beam on the longer arm, and the substance to be weighed is attached at a fixed point to the shorter arm by a hook or scale-pan. The longer arm is graduated, and the weight of the substance is known from the graduation at the point where the movable weight is, when the beam is at rest. Roman steelyard. Fig. 4.

In fig. 4, A is the knife-edge or fulcrum, P the weight movable along AB, C the point whence the substance, whose weight W is required, is suspended.

CAB being horizontal, let O be the point on the other side of A where P would keep the steelyard at rest when the weight W is away. The moment of the weight of the steelyard about A is therefore equal to $P.\ AO$.

Now let the weight W be attached, and let M be the place of P when equilibrium is obtained. Then, taking moments about A,

$$\begin{aligned} W.\ AC &= P.\ AM + \text{moment of weight of steelyard} \\ &= P.\ AM + P.\ AO \\ &= P.\ OM. \end{aligned}$$

Hence, $W = \dfrac{P}{AC}.\ OM.$

And, since P and AC are invariable, W is proportional to OM: O therefore is the point from which the graduation must be made. Thus, if P be at B_1 when W is 1 lb, and we take $OB_1 = B_1B_2 = B_2B_3 = \ldots$; then when P is at B_2, B_3, ... W will be 2, 3, ... lbs.

Principle of the lever.

62. The preceding cases of the lever are only special applications of the general investigation in Chap. III. In fact, any body movable about a fixed point and acted on by forces in the plane of that point may be considered a *lever*, and the principle of § 38 is often quoted as the *principle of the lever*.

63. *The wheel and axle.*

Wheel and axle. Fig. 5.

This machine consists of a circular drum or wheel, which is attached to a cylinder or axle, its centre lying in the axis of the cylinder and its plane being perpendicular to this axis. The whole system runs freely on this axis, which is fixed; and the power P acts by a string coiled round the wheel, and supports a weight W which hangs from a string coiled round the axle. The strings being perpendicular to the axis, and also to the radii of the wheel and axle respectively at the points where they become uncoiled, we have for the condition of equilibrium, by § 39, taking moments about the axis,

$P \times$ radius of wheel $- W \times$ radius of axle $= 0$.

Hence the mechanical advantage $\left(\frac{W}{P}\right)$ is equal to the ratio of the radii of the wheel and axle. Mech. adv.

Cor. Any number of wheels and axles may run on the same axis, and the condition of equilibrium will be that the sum of the products of each power into the radius of its wheel is equal to the corresponding sum for the weights and radii of the axles, the powers being all supposed to turn in the *same* direction and the weights all in the opposite.

64. *The pulleys.* Pulleys.

A pulley is a wheel running freely on an axis, which, passing through its centre, is fixed in a block by which the pulley is suspended and to which a weight may be attached. The circumference of the pulley is grooved to admit of a string passing over or under it. The pulley is said to be *fixed* or *movable* according as its block is so.

Single fixed pulley.

Let P, Q be the forces, applied at the ends of the string passing over the pulley. The whole system being smooth, the tension of the string is the same throughout (§ 10), and therefore,

$$P = Q.$$

No mechanical advantage is gained or lost.

65. *Single movable pulley supported by a string passing under it, the free portions of the string being parallel, and a weight attached to the block.* Single movable pulley Fig 6.

Let P be the force applied to the string on one side of the pulley; then, the whole system being smooth, the tension of the string is the same throughout, and P is therefore also the force applied to the string on the other side. There are then two parallel forces, each equal to P, supporting a weight W which acts vertically. Hence the strings must be vertical, and

$$W = 2\,P.$$

Here the mechanical advantage is 2.

Cor. 1. If one end of the string be attached to a fixed point, the same result holds good, for the tension must be the same throughout.

Cor. 2. If the weight w of the pulley, including the block, be taken into account, it may be supposed attached to the block, and we have

$$W + w = 2P.$$

First system. Fig. 7.

66. *First system of pulleys.*

A number of pulleys are attached to the same block, which supports a weight, and the same string passes round all the pulleys.

The portions of the strings between the pulleys are supposed to be parallel, and will therefore also be vertical as in § 65. Let P be the force applied to the string; P will then be the tension throughout, and the weight W is supported by as many parallel forces, each equal to P, as there are parallel portions of the string at the lower block; and the number of these portions is evidently double the number of pulleys at this block. Hence, if n be the number of movable pulleys,

$$W = 2\,n\,P;$$

and the mechanical advantage is $2\,n$.

Cor. If the weight of all the pulleys within the lower block be w, the weight of the block also being included, we may suppose this weight attached to the block, and

$$W + w = 2\,n\,P.$$

Second system. Fig. 8.

67. *Second system of pulleys.*

Each pulley hangs by a separate string, the last pulley supporting the weight; the free portions of all the strings are parallel, and therefore vertical.

Let A_1, A_2, A_3,... be the pulleys, n being the number of them; W, the weight supported at the last one; P, the power applied at the first string. Number the strings 1, 2, 3,... according to the pulley under which each passes. The tension of each separate string is the same throughout.

The tension of (1) is P; the weight supported at A_1 is double the tension of (1), and therefore $= 2P$, and this is the tension of (2).

The weight supported at A_2 is double the tension of (2) and therefore $= 2(2P) = 2^2 P$, and this is the tension of (3).

The weight supported at A_3 is double the tension of (3), and therefore $= 2(2^2 P) = 2^3 P$, and this is the tension of (4).

Proceeding in this way, we come at last to the weight supported at $A_n = 2^n P$, and this is the attached weight. Hence,

$$W = 2^n P,$$

and the mechanical advantage is 2^n.

Cor. 1. The mechanical advantage is doubled by every additional pulley.

Cor. 2. The weight of the pulleys may be readily taken into account by observing that, from the preceding, the force required to support a weight W on n movable pulleys is $\frac{W}{2^n}$. Pulleys supposed heavy

Let w_1, w_2, w_3, ... be the weights of the several pulleys, blocks included. Each of these weights may be supposed a weight attached to its block, and supported on the system of pulleys above it.

The power required to support w_1 on one movable pulley is $\frac{w_1}{2}$.

" " " w_2 on two " pulleys is $\frac{w_2}{2^2}$.

" " " w_3 on three " " $\frac{w_3}{2^3}$.

" " " w_n on n " " $\frac{w_n}{2^n}$.

Also " " " W on n " " $\frac{W}{2^n}$.

And the whole power required will be the sum of these; therefore,

$$P=\frac{w_1}{2}+\frac{w_2}{2^2}+\ldots.+\frac{w_n}{2^n}+\frac{W}{2^n}, \text{ or}$$

$$W=2^n P-\left(2^{n-1}w_1+2^{n-2}w_2+\ldots+w_n\right).$$

The weight of the pulleys, therefore, lessens the advantage of the machine.

Cor. If the weight of each pulley be the same (w), then

$$W=2^n P-\left(2^{n-1}+2^{n-2}+\ldots+1\right)w$$

$$=2^n P-\left(2^n-1\right)w.$$

Third system, Fig. 9.

68. *Third system of pulleys.*

Each pulley hangs by a separate string, which is attached to a bar or block carrying the weight, and the free portions of all the strings are parallel, and therefore vertical.

This is the second system turned upside down, the weight becoming a fixture, and the beam to which the strings are attached becoming a movable bar carrying a weight, and the mechanical advantage might be inferred from the preceding. The pressure supported by the beam in the second system is the sum of the tensions of the strings, that is,

$P+2P+2^2P+\ldots$ to n terms, $=(2^n-1)P$, and this becomes the weight W in the third system. Therefore,

$$W=\left(2^n-1\right)P.$$

The last pulley (A_n), however, becomes fixed, so that the number of movable pulleys is only $(n-1)$. Making n the number of *movable* pulleys, we have

$$W=\left(2^{n+1}-1\right)P.$$

The following is an independent investigation for this case.

Let A_1, A_2, A_3, ... A_n, be the pulleys, n being their number exclusive of the last one A, which is fixed, and $n+1$ the number of strings.

B_1, B_2, B_3, ... B_{n+1}, the points at which the respective strings are attached to the straight bar which carries the weight W. Number the strings 1, 2, 3, ... according to the pulley over which each passes.

The tension of each separate string is the same throughout.

The weight supported is the sum of the pressures of the strings at B_1, B_2, B_3. ...

The tension of (1) is P, and this is the pressure at B_1.

The weight supported at A_2 is double the tension of (1) and $= 2P$, and this is therefore the tension of (2) and the pressure at B_2.

The weight supported at A_3 is double the tension of (2) and $= 2(2P) = 2^2 P$; and this is therefore the tension of (3) and the pressure at B_3.

Proceeding in this way, we obtain the tension of the $(n+1)$th string and pressure at $B_{n+1} = 2^n P$.

Taking the sum of all these pressures,

$$W = P + 2P + 2^2 P + \ldots\ldots + 2^n P$$

$$= (2^{n+1} - 1) P,$$

and the mechanical advantage is $2^{n+1} - 1$.

Pulleys supposed heavy.

Cor. The weights of the pulleys may be taken into account by observing that each may be considered as a power acting by means of the string from which it hangs, and supporting a weight on the system of movable pulleys above it.

Let $w_1, w_2, w_3, \ldots w_n$, be the weights of the pulleys, blocks included.

The weight supported by w_1 on $(n-1)$ movable pulleys is $(2^n - 1) w_1$.

" " w_2 on $(n-2)$ " " $(2^{n-1} - 1) w_2$.

" " w_n on 0 " " $(2-1) w_n$.

Also " " P on n " " $(2^{n+1} - 1) P$.

The whole weight W (including that of the bar) is the sum of these; therefore,

$$W = P(2^{n+1} - 1) + w_1(2^n - 1) + w_2(2^{n-1} - 1) + \ldots + w_n(2 - 1).$$

The weight of the pulleys therefore increases the advantage of the machine.

Cor. 1. If the weight of each pulley be the same (w), then,

$$W = P(2^{n+1} - 1) + w(2^n + 2^{n-1} + \ldots + 2 - n)$$

$$= P(2^{n+1} - 1) + w(2^{n+1} - 2 - n).$$

If we put $P = 0$, we have

$$W = w(2^{n+1} - 2 - n),$$

which is the weight that would be supported by the pulleys alone.

Cor. 2. The point of the bar to which the weight should be attached, in order that the bar may be horizontal, will be the *centre of parallel forces* for the tensions of the strings and the weight of the bar. If we neglect the weight of the pulleys and the bar, this point will remain the same in a system, whatever be the power; if, however, the weight of bar and pulleys be considered, it will be different for different powers.

Systems compared.

69. Taking the same number n of movable pulleys in each system, the respective mechanical advantages are $2n$, 2^n, $2^{n+1} - 1$, and these numbers are in ascending order of magnitude. Hence the mechanical advantage of the third system is greater than that of the second, and of the second than of the first when there is more than one pulley.

Spanish Barton. Fig. 10.

70. The following combination of pulleys may be noticed. It is called the *Spanish Barton.*

The tension of the string to which P is attached is the same throughout and $= P$. That of the other string is also the same throughout and $= 2P$. Therefore $W = 4P$.

If we take the weights of the pulleys A, B into account, we have $W + B = 4P + A$.

71. *The inclined plane.*

Inclined plane.

This is a plane fixed at a certain angle (called its *inclination*) to the horizon, and on it a heavy particle is supported by a force applied and the reaction of the plane. Since the plane is smooth, its reaction is exerted in a normal direction; also the weight of the particle acts vertically: therefore, if a vertical plane be drawn through the particle and the normal to the inclined plane, since the plane thus drawn contains the directions of those two forces acting on the particle, the third force or *power* must also act in this plane.

Let the figure represent this plane; AB, the section of the inclined plane; AC, horizontal. Fig. 11.

The angle BAC is the inclination, α (suppose).

Let P, the power, act at an angle θ to AB, and let

R be the reaction of the plane exerted perpendicularly to AB.

W the weight of the particle acting vertically downwards.

The particle is then kept at rest by the three forces P, R, W.

Taking the resolved parts of these along AB, that of P is $P\cos\theta$; of R is 0; of W is $W\cos(90^\circ - \alpha) = W\sin\alpha$.

Hence by the "vanishing of the resultant,"

$$P\cos\theta - W\sin\alpha = 0,$$

which gives the mechanical advantage $\left(\frac{W}{P}\right) = \left(\frac{\cos\theta}{\sin\alpha}\right)$.

Cor. 1. For a given inclination, the mechanical advantage is greatest when $\cos\theta$ is greatest; that is, when $\theta = 0$, and the force acts parallel to the plane.

For a force acting at a given angle to planes of different inclinations, the mechanical advantage increases as the inclination diminishes.

Cor. 2. Resolving the forces horizontally, we have

$$P \cos(\theta + \alpha) - R \sin \alpha = 0.$$

Also resolving them perpendicularly to the direction of P,

$$W \cos(\theta + \alpha) - R \cos \theta = 0.$$

These two equations give R in terms of P or W.

Or these relations might at once have been asserted from the "triangle of forces" (§ 29): for this gives

$$\frac{P}{\sin \alpha} = \frac{W}{\cos \theta} = \frac{R}{\cos(\theta + \alpha)}.$$

Although these results have been obtained only for a particle, they are true for a body of finite size supported on the plane by a power whose direction passes through its centre of gravity.

Power acting parallel to plane. Fig. 12.

Stevinus.

72. In the case where the power is acting parallel to the plane, as where it is exerted by a string, parallel to the plane, passing over a pulley and supporting a weight P hanging freely, we have from the above by putting $\theta = 0$, or at once by resolving the forces along AB, observing that P is the tension of the string,

$$P - W \sin \alpha = 0.$$

and the mechanical advantage $\frac{W}{P} = \frac{1}{\sin \alpha}$, and is the cosecant of the inclination.

If BC (vertical) be called the *height* of the plane, AB its length: then, since $\sin \alpha = \frac{BC}{AB}$, we have

$$P - W \frac{BC}{AB} = 0,$$

or,

$$\frac{P}{W} = \frac{\text{height}}{\text{length}}.$$

Again, resolving forces perpendicularly to the plane, we have

$$R - W\cos\alpha = 0,$$

or $$R = W\cos\alpha = W\frac{AC}{AB};$$

and $$\frac{R}{W} = \frac{\text{base}}{\text{length}}.$$

Hence the power, weight and pressure on the plane are proportional to the height, length and base of the plane.

Cor. This latter result is at once seen from the "triangle of forces;" for, drawing CN perpendicular to AB, the sides of the triangle BCN, taken in order, are parallel to the directions of the forces, and therefore represent them in magnitude; and the triangle ABC is similar to BCN.

73. *The screw.* Screw.

The screw is a circular cylinder, on the surface of which runs a protuberant spiral thread, whose inclination to the axis of the cylinder is everywhere the same. This thread works freely in a fixed block, wherein has been cut a corresponding groove. The power is applied perpendicularly to a rigid arm which passes perpendicularly through the axis of the cylinder and is rigidly attached to it, and the weight is supported on the cylinder (whose axis is here supposed to be vertical), and may be supposed to act in the direction of this axis. Fig. 14

74. The complement of the invariable inclination of the thread to the axis, or (the axis being vertical) the inclination to the horizontal line which touches the cylinder at the point, is called the *pitch* of the screw. If a right-angled triangle BAC be drawn, having the base AC equal to the circumference of the cylinder, and the angle BAC equal to the pitch of the screw (α), and this triangle be wrapped smoothly on the cylinder, its hypothenuse will mark on the cylinder the course of the thread, and by superposing similar Fig. 13.

triangles, the whole course of the thread may be continued. BC is then the distance between two contiguous threads, and we have

$$\tan BAC = \frac{BC}{AC},$$

$$\text{or } \tan \alpha = \frac{\text{distance between two contiguous threads}}{\text{circumference of cylinder}}.$$

Fig. 14.

75. The screw is kept at rest by the weight (W) which acts vertically, by the power (P) which acts horizontally, and by the reactions of the groove on the thread at the various points in contact.

Since the thread is smooth, the reaction at each point of it is normal to the thread; and the angle between the directions of this normal and the axis, being the same as that between the thread and the horizontal tangent which are respectively perpendicular to them, is α, the *pitch*.

Fig. 15.

If then we resolve this reaction at any point, R (suppose), into two forces—one vertical, and the other horizontal and touching the cylinder—the former will be $R \cos \alpha$, and the latter $R \sin \alpha$.

All these vertical portions being parallel, will form a single vertical resultant whose magnitude is $\cos \alpha \ \Sigma (R)$, and this must counterbalance the weight W, since all the other forces are horizontal.

$$\text{Hence} \qquad \cos \alpha \ \Sigma (R) = W. \quad \ldots\ldots \qquad (1)$$

Again the horizontal portions of the R's tend to turn the cylinder about its axis, and since each acts in a horizontal direction touching the cylinder, the radius of the cylinder is itself the perpendicular distance between the axis and the direction in each case. Hence the moment of one of these ($R \sin \alpha$) is

$$R \sin \alpha \times \text{radius of cylinder},$$

and the sum of them all is

$$\sin a \times \text{radius of cylinder} \times \Sigma (R),$$

and this must be equal and opposite to the moment of the power, namely, $P \times$ arm of P.

Hence $\sin a \times$ radius of cylinder. $\Sigma (R) = P \times$ arm of P. Dividing the sides of this equality by those of the equality (1),

$$\frac{\sin a}{\cos a} \times \text{radius of cylinder} = \frac{P \times \text{arm of } P}{W}.$$

Hence, since $\frac{\sin a}{\cos a} = \tan a = \frac{\text{distance between threads}}{\text{circumference of cylinder}}$,

$$\text{dist. bet. threads} \times \left(\frac{\text{radius}}{\text{circum.}} \text{ of cylinder}\right) = \frac{P \times \text{arm of } P}{W};$$

$$\text{but } \frac{\text{radius}}{\text{circum.}} = \frac{1}{2\pi},$$

and $2\pi \times$ arm of P is the circumference of the circle which the end of P's arm would describe, and may be briefly called the circumference of P; hence

$$\frac{P}{W} = \frac{\text{distance between contiguous threads}}{\text{circumference of the power}},$$

and this ratio inverted is the mechanical advantage.

Cor. 1. The mechanical advantage is increased by diminishing the distance between the threads or increasing the arm of the power.

Cor. 2. If the cylinder be heavy, its weight must be included in W. Instead of supporting a weight, the screw may be producing a pressure at its lower end, and in this case the pressure produced will be increased by the weight of the screw. It may also be producing a pressure at its upper end, and then the pressure produced must be diminished by this weight.

Sometimes the screw is fixed and the block movable, as in the case of a common *nut;* or the groove may be cut in the screw itself, and the thread project in the block.

To all these cases the preceding investigation applies.

The wedge.

76. *The wedge.*

This is a solid, whose bounding surfaces are two intersecting planes. Most cutting instruments come under this class. It is used generally to separate the parts of bodies, either by blows or a moving pressure, and in this mode of use its investigation belongs to Dynamics. When used to keep open a rift in a body, it acts generally by means of friction, and not by a weight applied to it; it is therefore useless to proceed with its examination on the principles employed hitherto.

VIRTUAL VELOCITIES.

Virtual velocities.

77. If a machine at rest under a power P and a weight W be put in motion, so, however, that its geometrical relations are unaltered, the space described by the point of application of the power, estimated always in the direction of the power, is called the *virtual velocity* of the power: and similarly for the weight.

Principle of.

The principle of virtual velocities asserts that the product of P by P's virtual velocity is equal to that of W by W's virtual velocity.

78. This principle is only a special application of a far more general one, which it is not here necessary to examine. We shall therefore only establish the principle, as above

stated, in a few of the more simple cases, by proving that it leads to the relations of equilibrium already found.

The geometrical relations in a machine being such that the spaces described in different displacements are always proportional, it will be only necessary to prove the principle for a particular displacement, and we may select this as convenient.

79. *The straight lever under weights at its ends.* Straight lever.

Let the lever BAC be horizontal, and displace it round A into the position B_1AC_1 the directions of P and W meeting BAC in b, c. Then B_1b is P's *virtual velocity*, and C_1c is W's. Fig. 16.

The principle then asserts that

$$P.\ B_1b = W.\ C_1c;$$

but by similar triangles, $\dfrac{B_1A}{B_1b} = \dfrac{C_1A}{C_1c}$;

hence $P.\ AB = W.\ AC$, the condition found in § 57.

80. *Wheel and axle.* Wheel and axle

Suppose the machine to make one complete turn; then, the space descended by P is the circumference of the wheel; and by W, that of the axle. The principle then asserts that Fig. 5.

$P \times$ circumference of wheel $= W \times$ circumference of axle,

and the circumferences are as the radii: therefore,

$P \times$ radius of wheel $= W \times$ radius of axle,

the condition found in § 63.

81. *Pulleys.* Pulleys.

In the single fixed pulley, the principle is obviously true.

In the single movable pulley (fig. 6), let the pulley be raised through one inch, then W is raised through one inch, Single pulley.

and one inch of each portion of the string is set free; therefore, P ascends through two inches, and the principle asserts that

$$P \times 2 = W \times 1$$

which is the condition in § 65.

First system. Fig. 7.

82. In the first system of pulleys, let the lower block be raised one inch: then W is raised one inch, and one inch of each portion of the string at the block is set free; therefore, on the whole, $2n$ inches are set free, and this is the space through which P descends.

The principle then asserts that

$$P \times 2n = W \times 1,$$

the condition found in § 66.

Second system. Fig. 8.

83. In the second system of pulleys, let W be raised 1 inch: then A_n rises through 1 inch, and each pulley rises through twice as much as the one below it, and P rises through twice as much as the top pulley; therefore on the whole, P's ascent will be 2^n inches; and the principle asserts that

$$P \times 2^n = W \times 1,$$

the condition found in § 67.

Third system. Fig. 9.

84. In the third system of pulleys let W and the bar be raised 1 inch: then each pulley descends through this 1 inch and twice as much as the pulley above it, and P descends through 1 inch and twice as much as A_1.

The last pulley A_n descends through 1: therefore, the last but one descends through $1 + 2 \times 1$

" two " " $1 + 2\,(1 + 2)$ or $1 + 2 + 2^2$

" three " " $1 + 2\,(1 + 2 + 2^2)$ or $1 + 2 + 2^2 + 2^3$

and so on.

On the whole, $P = 1 + 2 + 2^2 + 2^3 + \ldots + 2^n$, or $2^{n+1} - 1$.

And the principle asserts that

$$P \times (2^{n+1} - 1) = W \times 1,$$

the condition found in § 68.

85. In the inclined plane, the power acting parallel to the plane, (fig. 12), let W be at the bottom of the plane and be drawn up to the top. Then W's vertical displacement is the height of the plane, and P's descent is its length. The principle asserts that Inclined plane. Fig. 12.

$$P \times \text{length} = W \times \text{height},$$

the condition found in § 72.

86. In the screw, let one complete turn be made. Then the distance moved through by the end of P's arm, estimated always in the direction of P, is the circumference of P; and the space descended by W is the distance between two threads. The principle then asserts that Screw. Fig. 13.

$P \times$ circumference of $P = W \times$ distance between two threads, the condition found in § 75.

87. Assuming the truth of this principle of *virtual velocities*, it may be conveniently employed to find the mechanical advantage in many machines—as examples, let us take *Roberval's balance*, *the differential axle*, and *Hunter's screw*.

88. In *Roberval's balance* the sides of a parallelogram are connected by free joints with each other and with a vertical axis passing through the middle points of opposite sides; so that the figure is symmetrical about this axis, and the other opposite sides are always vertical. The weights P, W are carried by arms fixed perpendicularly to these latter sides, which arms are therefore always horizontal. If the machine, when at rest, be displaced, one of the weights ascends as much as the other descends, and they are therefore equal. Roberval's balance. Fig. 17.

This result is independent of the particular points of the arms from which the weights depend, and in this lies the convenience of the machine as a balance.

Differential axle.

Fig. 18.

89. In the *differential axle* (fig. 18), two axles of different sizes run fixed together on the same axis, and the weight is supported on these by a pulley, whose string is coiled round these axles in opposite directions. If P be raised by a complete turn of the machine, W descends through a space equal to half the quantity of string set free from the axles; that is, through half the difference of the circumferences of the axles; and, the circumferences being as the radii, we have

$$P \times \text{radius of wheel} = W.\ \tfrac{1}{2}\ (\text{difference of radii of axles}).$$

In the common wheel and axle, the power and wheel being given, the mechanical advantage is increased by diminishing the radius of the axle; but this diminution is practically limited by regard to the strength of the axle. In the above machine, the mechanical advantage may be increased indefinitely, by making the axles more nearly of equal size, without too much weakening them.

If the axles were absolutely equal, the mechanical advantage would be infinite, and it is obvious that any weight would be here supported without a power at all.

Hunter's screw.

Fig. 19.

90. In *Hunter's screw* (fig. 19), the weight is supported on a smaller screw, which runs in a companion in the interior of a large screw, the latter passing through a fixed block and being acted on by a power as usual.

When the power makes a complete revolution, and raises the large screw through the distance between its threads, the smaller screw at the same time descends *in* the large one through the distance between its own threads, and the weight therefore on the whole rises through the difference between the "distances of the threads" in the two screws. Hence,

$P \times$ circumference of $P = W \times$ difference between distances of contiguous threads in the two screws.

The mechanical advantage can therefore be indefinitely increased by making the distance between the threads more nearly the same in each screw. In the common screw, the advantage is increased by diminishing the distance between the threads, but the diminution is practically limited by regard to the strength of the thread.

If each screw had the same distance of threads, the advantage would be infinite, and it is obvious that any weight would be supported without a power at all, the outer screw rising just as much as the inner screw descends *within* it, so that the weight would be stationary.

In every machine.

91. When a power P is supporting a weight W on any machine, if the machine be set in motion, it will continue to move uniformly so long as its geometrical relations with the power and weight are unaltered; and if s, S be the spaces gone through by the power and weight in any time (that is, their *virtual velocities*), we have $P \times s = W \times S$. Hence a given force acting through a given space for any time will lift the same weight only through a given space, whatever be the machine through which it acts; and if the weight lifted be increased, in the same proportion will the space through which it is lifted be diminished. Also when a given power lifts a weight through a given space, the greater the weight, the greater in the same proportion is the space through which the power must act, and (the motion being uniform) the longer is the time employed. Hence the principle of virtual velocities is sometimes stated in the form, that "in every machine, what is gained in power is lost in time."

what is gained in power

is lost in time.

Work done and efficiency.

92. Hence also this product $P \times s$ or $W \times S$ may be considered the *work done* by the machine, and is sometimes termed its *duty*; while with reference to the power, the names of *mechanical efficiency* and *laboring force* have been given.

In this sense, although *advantage* may be gained by a machine, no *efficiency* is gained or (*theoretically*) lost, but it remains the same as if the power were applied directly without the intervention of the machine.

33000 – lb. ft 1 m

Practically, *efficiency* is always lost, owing to the various resistances due to the parts of the machine.

Horse-power.

93. Among engineers the standard of *efficiency* in the comparison of machines has usually been taken to be a *horse-power*, which is represented by 33,000, a lb. and foot being the units employed, and the power being exerted for one minute of time. Thus a horse in one minute is supposed to lift 33,000 lbs. through 1 foot, or 3,300 lbs. through 10 feet, or 330 lbs. through 100 feet, and so on. A machine is then said to be of so many horse-powers, whence the *work done* by it in any time can be calculated.

FRICTION.

Friction.

94. Hitherto the surfaces of bodies in contact have been considered *smooth*, and exerting on each other no pressure except in a normal direction. In nature, however, all surfaces are more or less rough, and when one surface is pressing or moving upon another, a force is called into play which acts in a direction contrary to that of the motion, or to that in which motion would occur if the surfaces were smooth. This force is called friction.

Effect of in machines.

In machines, when a power is supporting a given weight, the magnitude of the power, determined on the supposition of the smoothness of the machine, may be increased beyond this value without disturbing the equilibrium, until it is great enough to overcome the friction together with the weight; and on the other hand, may be diminished till it is so small as with the aid of friction just to prevent the weight overcoming it. So also, with a given power, the weight may be increased or diminished within certain limits without disturbing the equilibrium. Generally, when the power is on the point of raising the weight, friction acts to the disadvantage of the power; but, when the power is just preventing the weight from descending, friction acts advantageously. When the equilibrium of a system depends on position, this position may with the aid of friction be varied within certain limits of the position determined on the supposition of smoothness, and the equilibrium be still maintained.

95. The motion of one surface upon another may be of the nature of *sliding*, or *rolling*, or both these. The former will be the case when two plane surfaces are in contact, and the laws of the friction in this case (denominated *sliding friction*) have been determined by experiment, the two surfaces, however, only *tending* to slide and not in actual motion. They are,— Sliding friction.

I. Between plane surfaces of given substances, the amount of friction is independent of the extent of area in contact, and depends only on the mutual pressure between them. Laws of.

II. The amount of friction is, for the same two substances, proportional to this normal pressure.

Hence, by the second of these laws, if F be the friction, and R the normal pressure, $\frac{F}{R}$ is a constant quantity for two given substances. It is called the *coefficient of friction* for these substances, and may be determined experimentally as follows: Coefficient of.

96. Let one of the substances form an inclined plane, (fig. 20), and a block of the other, of known weight W, and having a plane base, be placed upon it; and, by varying the inclination of the plane, let that inclination (α) be found at which W is just on the point of sliding down the plane. Found by experiment. Fig. 20.

Then F acts upwards along the plane, and we have (§ 72)

$$F = W \sin \alpha,$$

$$R = W \cos \alpha.$$

Therefore the coefficient of friction $\left(\frac{F}{R}\right) = \frac{\sin \alpha}{\cos \alpha} = \tan \alpha.$

The values of this coefficient for various substances have been found by experiment.

AN

ELEMENTARY TREATISE

ON

MECHANICS;

DESIGNED AS A TEXT-BOOK FOR THE UNIVERSITY EXAMINATIONS FOR THE ORDINARY DEGREE OF B.A.

PART II.

DYNAMICS OF A PARTICLE.

BY

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PREFACE TO FIRST EDITION.

The arrangement of this elementary work differs from that of most of the recent English writers on the subject, and is in the main the same as that employed by Professor Sandeman in his "Treatise on the motion of a Particle." Adopting in full the principles and method of that admirable treatise, I have attempted little more than to translate out of the language of the Calculus into ordinary algebra the investigations there given of the simpler cases of particle-motion.

For the reason stated in PART I., I have not added any *examples*, and have endeavoured to be as concise as possible in any explanations or illustrations that have appeared necessary.

UNIVERSITY COLLEGE,
TORONTO,
April 1, 1858.

CHAPTER I.

THE MOTION OF A PARTICLE GEOMETRICALLY CONSIDERED.

1. When the distance between two particles changes continuously during an interval of time, they are relatively in *motion.* Motion of a point

The position, and consequently the motion, of one particle can only be conceived in relation to other particles, but it is convenient to speak of a particle *absolutely* as being at rest or in motion, reference being made to ourselves or to some points in known relation to ourselves, considering these as *fixed*, and referring all motion and change of motion to the particle itself.

By a particle is here to be understood only a geometrical point.

Uniform motion.

2. When a particle is moving in a fixed straight line, its motion is measured by the change of its distance from a fixed point in this line, and the rate of this change of distance at any instant of *time* is called the *velocity* of the particle at that instant. In a straight line. Velocity

The change of distance in any time is here the linear space described by the particle in that time. If equal spaces are described in equal times, the change of distance in any given time is always the same, and the rate of this change, or the *velocity*, is said to be *uniform*, and is measured by the space described in a given time. Uniform, how measured.

Taking a foot and a second as the units of linear space and time, the velocity v of a particle moving uniformly will be measured by the number of feet described in one second.

The space described in 1 second being v, that in 2 seconds will be $2v$; in 3 seconds, $3v$; and, generally, in t seconds, tv: Space described in any time.

hence, if s be the space described in time t, with a uniform velocity v,

$$s = vt.$$

Apparently this formula is proved only for the case where t is a whole number of seconds; but, if t be fractional, we can always assume a unit of time such that the interval of time expressed by t shall contain a whole number of these units, and the formula can then be shewn to apply. Thus let n be a whole number such that nt is also a whole number T; and let $\frac{1}{n}$th of a second be taken as the unit, and V be the velocity referred to this unit. Then the time t being expressed in this unit by a whole number T, we have $s = T.V = nt\ V = vt$; for v being the space in one second or n units, is n times the space in one unit, that is, $= n\ V$. Hence the formula is general.

Velocity, + and −

3. Assuming some fixed point in the line of motion, if a be the distance of the particle from it at one instant, and s be the distance, estimated in the same direction, after the time t during which the particle has been moving uniformly with the velocity v, we shall have $s = a + vt$, or $s = a - vt$, according as the particle has been moving in the direction towards which s has been estimated *positively*, or in the opposite direction. Both these cases can be included in one formula by indicating oppositeness of direction of velocity by the algebraic signs + and −. Thus, fixing on one direction from the fixed point towards which when measured the distances are to be considered *positive*, a velocity in this direction will be *positive*, and in the opposite direction, *negative*.

Distance of a moving point from a fixed point in its line of motion after any time.

Hence, if a particle move during successive intervals of time with different uniform velocities, and a be the distance from the fixed point at the beginning of the time, s its distance at the end, then

$$s = a + \Sigma(vt),$$

where Σ denotes the algebraic sum of all the products corresponding to that within the brackets; and the particle will be on one or the other side of the fixed point at the end of the time according as s comes out from this expression *positive* or *negative*.

The whole *space described* will, however, be the numerical sum of these products, disregarding algebraic signs.

4. When a particle moves from a fixed point in a straight line with different velocities during successive equal intervals of time, each velocity continuing uniform throughout its interval, the distance of the particle from the point at the end of the time is the product of the time by the arithmetic mean of all the velocities.

Lemma.

Distance from starting point after a continuous series of uniform motions in the same line.

By the *arithmetic mean* of a number of quantities is meant their algebraic sum divided by the number of them.

For let t be the whole time; $\frac{t}{n}$ the duration of each interval; $v_1, v_2, v_3, \ldots$ the successive velocities during the first, second, third ... intervals. Then the required distance will be the algebraic sum of the space described with these velocities; that is, by § 2;

$$v_1 \cdot \frac{t}{n} + v_2 \cdot \frac{t}{n} + v_3 \cdot \frac{t}{n} + \ldots\ldots; \text{ or,}$$

$$\frac{v_1 + v_2 + v_3 + \ldots\ldots}{n} \cdot t. \qquad \text{Q. E. D.}$$

is the product of the mean velocity and the time.

The case of any of the velocities being in the opposite direction (and therefore accounted negative) is here included; the resulting sign of the algebraic sum determining on which side of the fixed point the particle is at the end of the time.

Accelerated motion.

5. When the velocity is not uniform, but changes during the motion, the velocity of the particle at any instant is measured by the space which it would describe in a unit of time, if it were to move uniformly during that unit with this velocity.

Varying velocity, how measured.

The rate of change of velocity at any instant (provided it be continuous) is called the *acceleration*.

Acceleration

If the change of velocity in a given time be always the same throughout the motion, the acceleration is said to be *uniform*, and it is measured by this change of velocity in a given time.

Uniform, how measured.

The change of velocity may be either an increase or decrease, and in the latter case the *acceleration* is in effect a *retardation*. The use of both terms is, however, rendered unnecessary by introducing the algebraic signs + and −; for a decrease is algebraically a *negative* increase, and thus a retardation is a negative acceleration; and when we speak of velocity being increased, added, or generated, we also include the case of velocity being diminished, subtracted, or destroyed.

The velocity after any period of uniform acceleration

Taking a second as the unit of time, the *acceleration* f, when the motion is *uniformly* accelerated, is the change of velocity in one second. Then $2f$ is the change in 2 seconds, $3f$ in 3; and generally tf in t seconds.

Hence, if u be the velocity at the beginning of the time t, and v be the velocity at the end of this time, we have

$$v - u = ft, \text{ or}$$

$$v = u + ft.$$

From rest.

6. If the particle started from rest at the beginning of the time, that is, if $u = 0$, then we have

$$v = ft.$$

Retardation

7. If the motion be uniformly retarded, f is to be taken negatively, and we have

$$v = u - ft.$$

The particle will be reduced to rest when $u - ft = 0$, or in a time $\frac{u}{f}$; and after this, the velocity will be accelerated in the opposite direction and by the same steps in a reverse order; till after a time $\frac{2u}{f}$, its value will be the same as at starting.

Acceleration uniform; to find the space described and the place of the particle after any time.

8. *When a particle moves with a uniformly accelerated motion from a fixed point in a straight line, to find the distance from the point after any interval of time.*

Let f be the uniform acceleration of the particle's motion; u be its velocity when at the fixed point; s the required distance from this point after a time t.

Let t be divided into n equal intervals. Then, by § 5, the velocities at the beginnings of these intervals are,

$$u,\ u + f\frac{t}{n},\ u + 2\,f\frac{t}{n},\ \ldots\ldots,\ u + \overline{n-1}\cdot f\frac{t}{n};$$

and the mean of these* is $u + \frac{\overline{n-1}}{2}\, f\frac{t}{n}$.

Hence, by § 4, if the particle moved uniformly during each interval with the velocity at the beginning thereof, the distance required would be

$$ut + \frac{n-1}{2} f\frac{t^2}{n}, \text{ or}$$

$$ut + \frac{1}{2} ft^2\left(1 - \frac{1}{n}\right).$$

Similarly, if the particle moved uniformly during each interval with the velocity at the end thereof, the distance required would be

$$ut + \frac{1}{2} ft^2\left(1 + \frac{1}{n}\right).$$

Between these two values the actual distance s always lies; but if we increase indefinitely the number of the assumed intervals, and diminish the duration of each, $\frac{1}{n}$ becomes indefinitely small, and each of the above quantities approaches to the same limit, which must therefore be the value of s. Hence,

$$s = ut + \frac{1}{2} ft^2.$$

Cor. 1. If the particle start from rest, then $u = 0$, and we have Motion from rest

$$s = \frac{1}{2} ft^2.$$

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* If a, l be the first and last of a series of n quantities in arithmetic progression, their sum $s = \frac{a+l}{2} \cdot n$.

Hence, the mean of them $\frac{s}{n} = \frac{a+l}{2}$, or the mean of the first and last.

Motion retarded.

Cor. 2. When f is positive, the velocity is continually increased, and s is the space described by the particle in the time t. But if f be negative, we have

$$s = ut - \frac{1}{2}ft^2;$$

and the distance of the particle increases till the time $\frac{u}{f}$, when it is momentarily at rest, the space described being $\frac{u^2}{2f}$. After this the particle moves back by the same stages in reverse order, its distance diminishing till the time $\frac{2u}{f}$, when it is again at its starting point. It then moves to the other side of the point, s becoming negative and being now given numerically by the formula $\frac{1}{2}ft^2 - ut$, and the whole space described in the time t being $\frac{u^2}{f} + \frac{1}{2}ft^2 - ut$.

Another investigation of the same. Newton's method.

9. The following is another investigation of the above proposition, after Newton's manner. Draw any line AK representing on any scale the number expressing the time t, and divide AK into equal parts in the points B, C, D..... Draw Aa perpendicular to AK, and take its magnitude on the same scale to represent the number expressing u the initial velocity. In the same way take Kk' to represent $u + ft$, the velocity at the end of the time. Draw ak parallel to AK, and at each of the points B, C, D...., draw perpendiculars to AK, meeting ak' in b', c', d'...., and complete the parallelograms in the figure. Then, since kk' represents ft, which is the change of velocity in the time AK; by similar triangles, dd' will represent the corresponding change in the time AD, and Dd' will represent the velocity at the end of this time, and similarly for each of the lines Bb', Cc', ...

Fig. 1.

Now, if the particle moved *uniformly* during any interval as CD with the velocity Cc', which it has at the beginning of this interval, the space described (§ 2) would be represented

numerically by the area of the inner parallelogram Cd; so also if it moved *uniformly* during CD with the velocity Dd', which it has at the end of this interval, the space described would be represented by the area of the outer parallelogram Cd'. If, therefore, the particle moved uniformly throughout each interval on the former supposition, the whole space described would be the sum of the inner parallelograms; and if on the latter supposition, it would be the sum of the outer parallelograms; and the space (s) actually described lies numerically between the spaces described on these two suppositions. But as the number of intervals is increased, and the magnitude of each diminished, the two series of parallelograms both approach nearer and nearer to the quadrilateral area $AKk'a$, and this must therefore be the value of s. Hence s is represented by the parallelogram Ak and triangle akk', that is numerically by

$$Kk \times AK + \frac{1}{2}\, kk' \times ak, \text{ and, therefore,}$$

$$s = ut + \frac{1}{2}\, ft^2.$$

Cor. 1. If the particle were at rest at the beginning of the time, that is, if $u = 0$, the line ak coincides with AK, and the space described is represented by the area of the triangle akk'. Hence,

From rest. Fig. 1.

$$s = \frac{1}{2}\, ft^2.$$

Cor. 2. If the velocity be retarded instead of accelerated, the figure will take another form. At first, the space described and the distance from the initial point at the time AK will each be represented by the area of $AKk'a$. At the time AL, the velocity will be destroyed and the particle momentarily at rest, the space described and the distance from the initial point being represented by the triangle ALa. Afterwards the particle returns towards its initial point, and the whole space described in the time AM will be represented by the sum of the areas of the triangles ALa, LMm', but the distance of the particle from the initial point will be represented by the difference of these two triangles, and at the

Motion retarded. Fig. 2.

time AN ($= 2AL$) this distance vanishes, and the particle is again at the initial point, the whole space described being represented by twice the triangle ALa. Afterwards the particle passes to the other side of the initial point, and its distance from it at the time AP is represented by the area $Nn'p'P$, while the whole space that has been described in this time is represented by the sum of the triangles ALa and LPp', that is, by twice the triangle ALa and the quadrilateral $Nn'p'P$. This result is identical with that in § 8, Cor. 2.

Motion from rest with uniform acceleration.

10. When the particle moves from rest and its motion is uniformly accelerated, we have seen that the velocity and space described at any time from the beginning of motion are given by the formulas,

$$v = ft;\ s = \frac{1}{2}ft^2,$$

and these are sufficient to determine all the circumstances of the motion in any case.

When any two of the quantities f, v, s, t, are given, the remaining two can be found from the above equations. The following cases may be noticed:

11. Given the acceleration and space described, to find the velocity acquired.

Here $s = \frac{1}{2}ft^2 = \frac{1}{2}f\left(\frac{v}{f}\right)^2$, and, therefore,

$$v^2 = 2fs;$$

and conversely, to find the space through which the particle must move to acquire a given velocity, we have

$$s = \frac{v^2}{2f}.$$

12. The equation $s = \frac{1}{2}ft^2$ becomes, by putting v for ft, $s = \frac{1}{2}vt$. Hence the space described in acquiring any velocity is half the space which would be described with that velocity continued uniform through the same time.

13. Putting $t = 1$, we have $s = \frac{1}{2}f$, or $f = 2s$. Hence, twice the space described in the first second from rest measures the acceleration.

14. *The spaces described from rest in successive equal intervals of time, are as the odd numbers*, 1, 3, 5, 7, ...

For, taking any interval as the unit of time, let F be the acceleration referred to it.

Then the space described in $\overline{n-1}$ intervals from rest is $\frac{1}{2}F(n-1)^2$, and the space described in n intervals from rest is $\frac{1}{2}Fn^2$.

The difference between these is the space described in the n^{th} interval, and $= Fn - \frac{1}{2}F = \frac{1}{2}F(2n-1)$.

Giving to n the successive values 1, 2, 3, ... this becomes $\frac{1}{2}F.1$, $\frac{1}{2}F.3$, $\frac{1}{2}F.5$, ... which was to be proved.

Circumstances of motion when the particle was not at rest at the commencement of the period.

15. The initial velocity being u, and this being uniformly accelerated during the time t, the velocity v at the end of this time and the distance s of the particle from its initial point, are given by the equations

$$v = u + ft;\ s = ut + \frac{1}{2}ft^2,$$

and these are sufficient to determine all the circumstances of the motion in any case.

When any three of the quantities u, f, t, v, s, are given, the remaining two can be found from the above equations.

The following cases may be noted:

16. We have $s = ut + \frac{1}{2}ft^2$

$$= \frac{1}{2}t(2u + ft)$$

$$= \frac{1}{2} t (u + u + ft)$$

$$= \frac{1}{2} t (u + v);$$

or, the distance is that which would be described in the same time with a uniform velocity equal to the mean of the initial and terminal velocities.

This result might at once have been inferred from § 8.

17. Given the initial velocity, the acceleration and the distance, to find the velocity acquired.

Here u, f, s are given to find v, and t must be eliminated from the two equations

$$v = u + ft, \; s = ut + \frac{1}{2} ft^2.$$

Squaring the first, we have

$$v^2 = u^2 + 2uft + f^2 t^2$$

$$= u^2 + 2f (ut + \frac{1}{2} ft^2)$$

$$= u^2 + 2fs.$$

If the velocity were retarded, we should have

$$v^2 = u^2 - 2fs.$$

Cor. This result might have been obtained without finding the second equation, for we have directly, from § 5,

$$v - u = ft,$$

and from § 16 or 8,

$$\frac{1}{2} t (v + u) = s;$$

multiplying these equalities, we have

$$v^2 - u^2 = 2fs.$$

The following geometrical proof may also be noticed :

Let B be the initial point, where the velocity is u ; BC the space described (s) when the velocity is v.

Let A be the point from which the particle, proceeding from rest under the same acceleration, would acquire the velocity u at B. Then (§ 11),

$$u^2 = 2\,f.\ A\,B.$$

Also, since the whole motion may be taken to proceed from rest at A, we have (§ 11)

$$\begin{aligned} v^2 &= 2\,f.\ A\,C \\ &= 2\,f.\ (A\,B + A\,C) \\ &= 2\,f.\ A\,B + 2\,f.\ A\,C \\ &= u^2 + 2\,fs. \end{aligned}$$

By a proper adaptation of the figure, this proof may be extended to all the cases included in the algebraic formula.

18. Give the initial velocity and the acceleration, to find the time when the particle will be at a given distance from the initial point.

Here u, f, s are given to find t.

Solving as a quadratic in t the equation $s = u\,t + \frac{1}{2} f t^2$, we have

$$t = \frac{-u \pm \sqrt{u^2 + 2\,fs}}{f}.$$

The significance of the double sign is here note-worthy.

If f be positive, or the velocity be numerically accelerated, one of the values of t is positive, and the other negative. The former is the solution required, but the latter can be interpreted thus: Suppose A the initial point, $A\,P$ the distance s, and the velocity u at A to be in the direction AP. Then the positive value of t in the above gives the time of moving from A to P; the negative value gives the time that would have elapsed if the particle had moved from P towards A, with a retarded motion, passed through A to the other side of it, been reduced to rest, and again returned to A.

If f be negative, then, writing $-f$ for f, the values of t become

$$\frac{u \pm \sqrt{u^2 - 2\,fs}}{f}.$$

If then $u^2 > 2fs$, both values of t are real and positive, and the particle will twice be at the same distance from the initial point, once during the recess from and again during the return towards it.

If $u^2 = 2fs$, the two values become the same, and the distance in question is that where the particle momentarily comes to rest.

If $u^2 < 2fs$, both values of t are imaginary, and the particle can never reach that distance.

If, however, s be negative, both values are real, and one positive, the other negative, the latter referring to a time previous to the epoch from which we are reckoning, when the particle, if it had been moving towards the initial point from the negative side, would have been at the assumed distance.

Component velocities.

Composition of velocities.

19. The position and motion of a particle moving uniformly in a straight line have been determined by the distance of the particle from a fixed point in the line, and by the change of this distance in a given time. Its position, however, might have been defined by its distances from two fixed lines, measured parallel to these lines. Thus: let Ox, Oy be two fixed lines, B the place of a particle moving in the line ABC, and A a fixed point in this line, the distance from which determines the place of the particle. Let C be the place at which the particle would arrive after any time if it moved uniformly with the velocity it had at B, and complete the figure by drawing lines parallel to Ox, Oy.

Fig. 3.

The position of B is known when BP, BQ, its distances from these fixed lines, are given; and CE, CD, or their equals, BD, BE, would be the changes of these distances if the particle arrived at C by moving uniformly.

Now BC, which would be the change of distance in a given time from the fixed point A, measures the velocity of

the particle: and BD, BE are *always* proportional to BC, and therefore measure what we may call the *component velocities* of the particle in the directions of the fixed lines. Hence,

Parallelogram of component velocities.

If a straight line be taken to represent in magnitude and direction the velocity of a particle, the adjacent sides of any parallelogram constructed on this line as diagonal will represent the COMPONENT VELOCITIES *in the directions of those sides.*

Conversely.

If the COMPONENT VELOCITIES *in two directions be given, the actual velocity will be found in magnitude and direction by drawing the diagonal of the parallelogram of which the components form adjacent sides.*

These two statements constitute the "parallelogram of component velocities."

Velocity resolved in any direction.

20. When the two components are in perpendicular directions, it will be convenient to call them the *resolved parts* of the velocity in these directions; and the rule for finding these resolved parts will be the same as that for the resolved parts of a force (STATICS, § 21), namely:

Rule for.

To find the resolved part of a velocity in any direction, *multiply it by the cosine of the angle between this direction and that of the velocity*; and to find the resolved part perpendicular to this direction, *multiply by the sine of the aforesaid angle.*

CHAPTER II.

THE MOTION OF A MATERIAL PARTICLE ACTED ON BY UNIFORM FORCES.

Application of preceding results to the actual motions of material particles.

21. In the foregoing chapter the geometrical conditions of the motion of a point have been examined. It now remains to exhibit the connection of these results with the actual motions of material particles, and the relation between these motions and the forces acting on the particles, and this investigation constitutes the science of *Dynamics*.

Experimental laws.

For this purpose it is necessary to appeal to experiment and observation, and it appears that all the phenomena of the motions of material particles can be referred to three elementary principles or laws, which are commonly known as "Newton's Laws of Motion." These laws, from their nature, are incapable of being demonstrated by direct experiment, for it is impossible to make experiments under the precise circumstances conditioned by the laws, and which would not involve other phenomena besides those which it is desired to test. Direct experiments may, however, afford a presumption in favour of these laws by showing that the more nearly do the circumstances of the experiment approach to the exact conditions required, the more nearly are the results of the experiment in accord with those indicated by the laws; and also that whenever a discrepancy is found between these results, there can always be traced some disturbing cause which ought to have been excluded by the conditions postulated.

The ultimate ground on which these and all other laws in natural philosophy rest, is the entire and universal concordance of the results of experiment or observation with those calculated on the assumption of the truth of the laws.

22. Although the motion of a particle and the forces acting on it can only be conceived in relation to other particles, it is convenient to speak *absolutely* of a particle as being at rest or in motion, reference being made to ourselves or to some space in a known relation to ourselves which we consider *fixed*, and then to regard the phenomena exhibited by the particle as due to forces acting only on itself, these forces being defined by the measures of them already employed in Statics.

23. First Law of Motion.*

Newton's three laws of motion.

First law.

A material particle, when not acted on by any force, if at rest, will so remain; and, if in motion, will move in a straight line with uniform velocity.

The first part of this law has been already assumed (Statics § 2) as the basis of our conception of a force. Experience shows that whenever a quiescent body is set in motion, we can trace the action of some cause external to the body; thus, when a body is suffered to drop to the earth, we assign its motion to a pressure exerted on it due to the earth itself, and which would have no existence if the earth did not exist. Also, there seems no reason why a particle, apart from any external force, should begin to move in one direction rather than another.

No forces acting,

the particle either remains at rest, or

Again, when a particle is in motion there seems no reason why it should change the direction of its motion in one way rather than another, unless some force be acting upon it to determine such change; and in all cases of any such change, we can always trace the action of some external force; as, for instance, when a stone is projected from the earth in any direction, the deflection of its motion from a straight line is produced by the aforesaid pressure due to the earth, which we know is always acting vertically downwards. If this pressure be counteracted by projecting the stone hori-

moves in a straight line

* Lex. I. *Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus a viribus impressis cogitur statum illum mutare.*—Princ. Leg. Mot.

zontally along a fixed plane, the path approaches to a straight line, with only such deviations as may be accounted for by friction or irregularities in the plane, or from the stone not being small enough to be considered a particle.

with uniform velocity.

Galileo.

So also with regard to the velocity of the particle: it does not seem possible to conceive any way in which its velocity could increase or decrease unless by the action of some external cause, and in actual cases of variation of velocity, we can always trace the existence of such causes. Thus when a stone is thrown horizontally along the ground, it gradually loses its velocity and stops, but here the friction of the ground and the resistance of the air act as retarding causes, and we see that in proportion as the surface on which the stone moves is smoother, as on a sheet of ice, the longer and more uniform does the motion continue.

This law is sometimes termed the *Law of Inertia*, being understood to express that a material particle is *inert*, and has no tendency of itself to change its state of rest or uniform motion.

24. It follows that the motion of a material particle, when not acted on by any forces, or acted on only by forces which counterbalance, is determined by the formula of uniform motion, $s = vt$, investigated in § 2.

Uniform forces acting on a particle.

25. We now proceed to consider the motion of a particle acted on by any uniform forces, of which the following are the observed laws :—

(1.) *When a uniform force acts continuously upon a particle in the line of its motion, the velocity is uniformly accelerated.*

A single force in the line of motion.

The investigations of § 5 *et seq.*, therefore, apply to this case, (noticing also that *retardation* is included in the term *acceleration*), and we can compare the results there calculated with those of experiment. Thus when a body is permitted to drop freely to the earth, or is projected vertically downwards or upwards with any assigned velocity, its path is a vertical line, and the force acting on it is its weight which

always acts vertically, and (for not great heights above the surface) is sensibly uniform. Here then the required conditions are fulfilled, and the result of experiment, when due allowance is made for the resistance of the air, is that the motion is uniformly accelerated, the amount of this acceleration being about 32·2 feet a second, but varying slightly for different latitudes and elevations above the sea-level. This acceleration is usually denoted by g.

Any forces in the line of motion.

(2.) *When several uniform forces are acting simultaneously in the line of motion, the resulting acceleration is the algebraic sum of the accelerations which would be produced by each force acting separately.*

Hence it follows that n equal forces acting simultaneously on a particle in its line of motion will produce n times the acceleration which one of the forces alone would produce on the same particle; and, consequently, the acceleration produced in a given particle is proportional to the magnitude of the force acting.

It will be hereafter shown how this may be tested by comparing the accelerations of a particle down inclined planes of different inclinations.

Hence also the change of velocity in a given time is proportional to the magnitude of the force, the particle acted on being the same.

A single force oblique to the direction of motion.

(3.) When a moving particle is acted on continuously by a uniform force which acts always in the same direction and obliquely to the direction of the particle's motion, its velocity after any time is found to have for components—first, the original velocity, unaltered, in its own direction; second, a velocity in the direction of the force, the magnitude of which is the same as if the force had acted on the particle originally at rest. So that the velocity and direction of the motion may be found at any time by calculating the velocity which would be produced by the force acting for that time on the particle originally at rest, and then compounding this with the original velocity according to the principle of the "parallelogram of velocities."

Or, this may be expressed more simply thus: if we resolve the original velocity of the particle into two components, one in direction of the force, the other perpendicular to it, the latter remains unaltered, and the former is changed by the force precisely as if it alone were the actual velocity of the particle. So that

The change of velocity produced by the force in a given time is in direction of the force, and is proportional to it in magnitude.

In this case the path of the particle is no longer a straight line, but a curve, the tangent to which at any point is the direction of the particle's motion there.

In other words, the above expresses that *the dynamical effect of a force on a particle is wholly independent of any motion which the particle may have, and is the same as if it were exerted on the particle originally at rest.*

Thus, the vertical descent of a body let fall from the mast-head of a ship in motion is precisely the same in all its circumstances as if the ship were at rest. The principle can also be tested by comparing the results of calculation with observations on the motion of a body projected obliquely to the horizon and acted on by gravity, due allowance being made for the resistance of the air.

Any forces acting in any direction on a particle either originally at rest

(4.) When several forces act simultaneously, retaining always the same magnitudes and directions, on a particle originally at rest, the motion is uniformly accelerated in the direction of the resultant of the forces, and the acceleration is that due to this resultant acting singly.

Also the velocity generated after any time, being that due to this resultant, is also that which is compounded of the velocities due to the forces acting singly on the particle from rest.

or in motion.

Also if the particle be in motion when the forces begin to act, its velocity and direction of motion after any time will

be determined by compounding its original velocity with the velocity due to the resultant of the forces, or with all those due to the forces separately. Or,

When forces act on the same particle under any circumstances provided each force be uniform and always preserve the same direction, the change of velocity in a given time due to each force is in direction of that force, and is proportional to it in magnitude.

(5). It follows from the preceding that if f be the acceleration due to a force P acting on a certain particle, then the ratio $P : f$ is invariable for this particle. This ratio is found, however, to be different in different particles, and we thus discover a quality which distinguishes one particle from another, of which this ratio will serve as a measure. The name of *mass* is given to it, and one particle has the same *mass* as another when the same force produces in each the same acceleration. The unit of mass is arbitrary and it is not necessary to fix it, but we shall take as the measure of *mass* the above ratio of the numbers expressing a force and the acceleration it produces on the particle. Thus, if m be the *mass* of a particle, and P, f as above,

For the same particle the acceleration is proportional to the force.

Mass defined

and measured

$$\frac{P}{f} = m.$$

It has been mentioned that the acceleration produced by gravity (g) is the same for all bodies at the same place on the Earth's surface. Hence, if W be the weight of a body, m its mass, we have

$$\frac{W}{g} = m, \text{ and}$$

$$W = m\,g.$$

Hence, for a given place, the weights of bodies may be taken to measure their masses.

The fact above stated (namely, that the acceleration of gravity is the same for all bodies at the same place) is apparently contradicted

Lucretius.

by the different times occupied by different bodies in falling from the same height to the earth; but this is due to the different resistances of the air, as is shown by trial in an exhausted receiver, where the feather and the guinea are seen to fall precisely in the same time.

(6.) Since $P = m f$, and f is proportional to the change of velocity in a given time (§ 5), it follows that P is proportional to the product of the mass and the change of velocity in a given time.

Momentum

The product of the mass and the velocity (that is, of the numbers expressing these,) is called the *momentum* of the particle, and the preceding results can now all be combined in one statement, which constitutes

THE SECOND LAW OF MOTION.*

Second law.

When uniform forces act continuously for a given time on material particles, each produces in its own direction a change of momentum proportional to itself in magnitude.

Impulsive forces.

26. The forces hitherto treated of have been of such a nature as only to produce finite changes in the motion of a particle by acting on it for a finite time. There is, however, a certain class of forces, such as those manifested in explosions or the collision of bodies, which produce finite effects in changing the velocity or momentum *instantaneously*. Such forces are called *impulsive*, and must be carefully distinguished from forces of the former class, with which they do not in any way admit of comparison. These impulsive forces are measured by the momentum which each would instantaneously communicate to a particle at rest, and the second Law of Motion applies to them, stated under the form:

Second law applied to.

When impulsive forces act on material particles, each produces in its own direction an instantaneous change of momentum proportional to itself in magnitude.

* Lex. II.—*Mutationem motus proportionalem esse vi motrici impressæ, et fieri secundum lineam rectam qua vis illa imprimitur.*—Princ. Leg. Mot.

Thus the motion of a particle when acted on by simultaneous impulses will be determined by calculating the velocity instantaneously generated by each in its own direction, and compounding these with the original velocity of the particle according to the parallelogram of velocities. For instance, if a particle at rest be acted on by two impulses which, separately communicated, would give the particle respectively such velocities as would cause it to describe uniformly the sides AB, AC of a parallelogram $BACD$, the particle will acquire from the impulses simultaneously communicated a velocity which will cause it to describe uniformly the diagonal AD in that time.

Parallelogram of velocities.

Aristotle.

APPLICATIONS AND TESTS OF THE SECOND LAW OF MOTION.

Vertical motion by gravity.

27. *The vertical motion of a particle under the action of gravity.*

Galileo.

The acceleration of gravity (g) has already been stated to be about 32·2 feet per second, and to be sensibly the same for all bodies in the same latitude and at nearly the same height above the sea-level.

$g = 32{\cdot}2$.

Hence, applying the formulas in § 10, we have, when the particle moves from rest,

From rest.

$$v = gt = 32{\cdot}2 \times t;\ s = \frac{1}{2}gt^2 = 16{\cdot}1 \times t^2.$$

Thus the spaces described from rest after the lapse of 1, 2, 3, ... seconds, are 16·1, 64·4, 144·9, ... feet; and the velocities acquired are 32·2, 64·4, 96·6, ... feet per second.

If the body do not fall from rest, but be projected downwards with a velocity u, then we have § 15

Projected down

$$v = u + gt,\ s = ut + \frac{1}{2}gt^2.$$

Or if it be projected vertically upwards with a velocity u, then the velocity and distance from the point of projection at the time t are given by

r up.

$$v = u - gt,\ s = ut - \frac{1}{2}gt^2;$$

and the particle is brought momentarily to rest after having ascended a height $\frac{u^2}{2g}$ in the time $\frac{u}{g}$, and then descends again by the same steps in reverse order, reaching its point of projection in the time $\frac{2u}{g}$.

The resistance of the air and the rapidity of the motion render it difficult to test these results directly by experiment.

Motion down an inclined plane.

Galileo.

28. *Motion down a smooth inclined plane.*

Let a be the inclination of the plane.

The particle is acted on by two forces, namely, its own weight (W) acting vertically, and the reaction of the plane in a normal direction. If we resolve W into two forces, one perpendicular to the plane, and the other ($W \sin a$) downwards along the plane, the motion estimated along the plane will be due to this latter only. Hence, f being the acceleration along the plane, we have § 26 (5),

$$\begin{aligned} f &= \text{Force} \div \text{mass of particle} \\ &= W \sin a \div \frac{W}{g} \\ &= g \sin a . \end{aligned}$$

And with this value of f the formulas of §§ 10 and 15 avail to determine fully the motion.

L. da Vinci.

Cor. For the same particle, on planes of different inclinations, the accelerations are as the sines of the inclinations, or, the length of the plane being given, as the heights; and this is the test mentioned in § 25, (2), allowance being made in performing the experiment for the resistance of the air and imperfect smoothness.

29. *The velocity acquired by moving from rest down an inclined plane, is equal to that acquired by falling freely down the height of the plane.*

For, f the acceleration down the plane is $g \sin a$, and if v be the velocity acquired in moving down its length, § 11,

$$\begin{aligned} v^2 &= 2f \times \text{length} \\ &= 2g \sin a \times \text{length} \\ &= 2g \times \text{height}; \end{aligned}$$

and this is the same as if the body fell freely down this height.

Cor. If the particle were projected down or up the plane with a velocity u, and v be the velocity after moving through any *length* of it, we should have in like manner, § 17,

$$\begin{aligned} v^2 &= u^2 \pm 2g \sin a \times \text{length} \\ &= u^2 \pm 2g \times \text{height}, \end{aligned}$$

which is the same as if the particle were projected vertically downwards or upwards with a velocity u, and moved freely through the corresponding *height*.

30. *The time of moving from rest at the highest point of a vertical circle down any chord (considered a smooth inclined plane) is the same as the time of falling freely from rest down the vertical diameter; and so is also the time of moving from rest down any chord to the lowest point.*

For, AB being the vertical diameter, the acceleration down the chord AC is $g \cos BAC$, and therefore, § 10, Fig. 4.

$$(\text{time down } AC)^2 = \frac{2\,AC}{g \cos BAC} = \frac{2\,AB}{g},$$

which is the square of the time down AB.

So also, the acceleration down CB is $g \cos CBA$, and

$$(\text{time down } CB)^2 = \frac{2\,CB}{g \cos CBA} = \frac{2\,AB}{g},$$

the same as in the former case.

A particle projected in any direction and acted on by gravity.

Galileo.

31. *Motion of a projectile.*

Let a particle be projected from a point in the horizon, with a velocity v, and in a direction making an angle a with the horizon. The force acting on it being its weight which always is directed vertically downwards, the motion of the particle will be in one vertical plane.

If we resolve its velocity of projection into two: namely, $v \cos a$ horizontally, and $v \sin a$ vertically; the former continues unaltered, and the latter is retarded and accelerated by gravity precisely as if the particle had been projected vertically with this velocity. Hence, g being the acceleration by gravity, the velocity $v \sin a$ is destroyed by it in a time $\frac{v \sin a}{g}$, (§ 10); at this moment the particle is moving horizontally, and has reached its greatest elevation above the horizon. The velocity $v \sin a$ is again generated by gravity by the same steps in a reverse order, till on again reaching the horizon the velocity is the same in magnitude, and its direction is equally inclined to the horizon in an opposite direction, as at the point of projection. The path, therefore, consists of two equal and similar branches on each side of the greatest elevation.

Time of flight.

The whole *time of flight* is therefore $2.\frac{v \sin a}{g}$, and during this time the horizontal distance described with the uniform velocity $v \cos a$ is (§ 2)

$$v \cos a. \quad 2.\frac{v \sin a}{g}, \text{ or}$$

$$2\frac{v^2}{g} \sin a \cos a, \text{ or}$$

$$\frac{v^2}{g} \sin 2a. \quad \text{(Trig. § 38.)}$$

Range.

and this is called the *range*.

The greatest elevation is the space due to the velocity $v \sin a$ for the acceleration g: that is (§ 11),

Greatest height.

$$\frac{(v \sin a)^2}{2g}.$$

Again, if x be the horizontal distance of the particle from the point of projection at the time t, and y its elevation above the horizon at that time, we have (§§ 2 & 15), Place at any time.

$$x = v \cos a.\ t,$$
$$y = v \sin a.\ t - \tfrac{1}{2} g t^2;$$

which determine the place of the particle at any time.

The path of the particle is the curve called by geometers a parabola. In comparing these results with observation, the resistance of the air has to be taken into account; and for large bodies, or considerable velocities, these results are thereby rendered quite wide of the observed facts.

32. Third Law of Motion.*

When one material particle acts on another, the second exerts on the first an action equal in amount and opposite in direction to that which the first exerts on it. Third law.

The actions here spoken of may be of various kinds; such as the mutual pressures between bodies in contact whether at rest or in motion; or the action of one particle on another by means of a stretched string or a rigid rod; or the action may be of the nature of attraction or repulsion; or finally of an *impulsive* character, as in cases of collision.

The measures of these actions are either their statical measures or those furnished by the second law of motion.

Sometimes the law is stated in the form:

The actions of bodies are mutual, equal, and opposite.

The law may be tested by direct experiment in various ways; such as by noting the motion of two bodies hanging freely by a string passing over a pulley: by observing the motion of a magnet and piece of iron floating on water: and by observing the velocities of balls that have suffered collision.

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* Lex. III. *Actioni contrariam semper et æqualem esse reactionem: sive, corporum duorum actiones in se mutuo semper esse æquales et in partes contrarias dirigi.*—Princ. Leg. Mot.

Motion of two weights over a pulley.

Newton.

33. Two bodies connected by an inextensible string, which passes over a smooth fixed pulley, descend by the action of gravity, to determine the motion.

Let P, Q be the weights of the two bodies, P being the greater of the two.

The pulley being smooth, and the weight of the string insensible, the tension of the string is, by the third law, the same on each body.

Since the string is supposed inextensible, the downward motion of P is the same as the upward motion of Q, and we may consider them as one mass acted on in the direction of motion by the uniform pressure $P - Q$.

The weight of the mass moved is $P + Q$, and the mass is therefore $\frac{P+Q}{g}$.

But the pressure divided by the mass is the acceleration (f); hence,

$$f = (P - Q) \div \frac{P+Q}{g}$$

$$= \frac{P-Q}{P+Q} g,$$

and with this value of f, the formulas of §§ 10 & 15 apply.

Atwood's machine.

Fig. 5.

34. It has been mentioned that the velocity of a body falling freely is too rapid to be conveniently experimented on. In the above case of motion, however, the acceleration can be made sufficiently small, depending as it does on the difference between P and Q, to enable observations to be made with some accuracy. This is effected by the arrangement known as *Atwood's Machine*, which consists essentially of two weights connected by a thin string passing over a pulley, and the disturbance caused by friction is lessened by the axle of the pulley being made to rest on friction-wheels. The resistance of the air, however, still interferes with the results.

The weights are contained in two boxes A, B, and the motion of the descending one A is measured by means of a vertical rod, graduated in inches, on which a movable stage can be fixed at any point, and the coincidence of the striking of the bottom of the box on this stage with the beat of a second's pendulum attached to the machine is employed to measure the time.

The weights used are equal pieces of brass, denominated *ems*, and it is found that the effect of friction and of the motion of the pulley can be represented by supposing an additional number of these *ems* to be added to the whole weight.

Thus, the weight of the two boxes being 12, and each being loaded with 20 *ems*, in which condition they would balance, let 1 *em* be added to A, and 11 *ems* be allowed for friction and the effect of the pulley.

Then the weight of the whole mass moved $(P+Q)$ is taken as 64 ems, and the moving pressure $(P-Q)$ is 1 em. Therefore, applying the formula in the preceding article,

$$f = \frac{1}{64} g.$$

On making the experiment the spaces described in 1, 2, 3, 4 seconds respectively are found to be about 3, 12, 27, 48 inches. And applying the formula $s = \frac{1}{2} f t^2$, the value of g comes out in each case 32 ft. per second.

The experiment may be varied in several ways, and we have thus a test not only of the third law, but of the uniformity of the acceleration produced by gravity, and of its constant value for bodies of different weights, § 25 (1). The machine may also be employed to test the proportionality of the acceleration to the moving pressure, § 25 (2), the mass moved remaining the same.

Thus, the boxes being loaded so as to balance, one or more *ems* in the form of long bars can be placed on A as a moving pressure, and a ring-stage can be fixed at any point of the

vertical bar which permits the box to pass freely, but removes the long *ems*. After which removal the box continues to descend uniformly with the velocity acquired (or would do so but for friction and the resistance of the air), and the space thus described in one second can be measured, giving the velocity acquired, and therefore the acceleration.

For example, let the boxes be loaded each with 20 ems; their weight 12; allowance for friction, &c., 11; and let 1 long em be added to *A*. Then the weight of the whole mass is 64. After the box has descended from rest through one second, let the long *em* be removed by the ring stage. Then the space described in the next second by the box moving uniformly is found to be six inches, and this is the measure of the velocity acquired in the first second, and therefore of the acceleration.

Hence, the mass being 64, and the moving pressure 1, the acceleration is 6.

Again, let the boxes be loaded each with 19 *ems*, and 3 long *ems* be put on *A*; and by the same method the acceleration is found to be 18.

Thus, the mass being 64, and the moving pressure 3, the acceleration is 18.

So that the acceleration is proportional to the pressure.

Again, the effect of gravity as a retarding force may be exhibited by allowing box *A* to acquire a certain velocity in its descent, then removing the long *ems*, so that the other box becomes the heavier, and noting the time when they are reduced momentarily to rest.

Impact and Collision.

Collision of smooth balls.

Newton.

35. Since balls are extended bodies, we cannot apply to them directly the laws for the motion of mere *particles*, but when the balls are uniform in substance, the motion of their centres will be the same as that of imaginary particles of the same masses as the balls, placed in these centres.

Direct impact of two balls.

36. The impact is said to be direct when the centres of the two balls are moving in the same line. The impulsive action which occurs between the balls on collision takes place wholly in this line (the balls being smooth), and is measured (§ 26) by the instantaneous change of momentum in each ball in this direction, and, by the third law of motion, this must be the same in amount and opposite in direction on each ball. That is, the gain of momentum by one ball is the same as the loss by the other, and the (algebraic) sum of the momenta of the two balls remains unaltered by the impact; or, in other words,

Two balls impinge directly.

The algebraic sum of the momenta after impact is equal to that before the impact.

Law of equality of momenta.

This gives one relation between the velocities after impact; but, to determine them, another relation is still required. This is furnished by the following experimental law:

The two balls either proceed in contact with a common velocity, or they separate in such a manner, that the magnitude of their relative velocity after impact bears to that of their relative velocity before impact a ratio which depends only on the nature of the substance of which the balls are composed.

Law of relative velocities.

In the former case the balls are called *inelastic;* in the latter, *elastic;* and the constant ratio above spoken of is called the *elasticity* for each particular substance, and is generally denoted by the letter e. Its value for all known substances lies between 0 and 1; thus for steel it is $\frac{5}{9}$; for ivory, $\frac{8}{9}$; for glass, $\frac{15}{16}$. If e were equal to 1 for any substance, it would be *perfectly* elastic, but no such substance is known in nature. It is clear that the case of inelastic bodies is included in that of elastic ones by giving to e the value 0.

Elasticity

By the *relative velocity* of the balls is meant the algebraic difference of their velocities. Thus if u, v are the velocities in the same direction before impact, and the ball moving with

u overtakes the other; then $u - v$ is their relative velocity and if u', v', are the corresponding velocities after impact, the second ball moving away from the first, then $v' - u'$ is the relative velocity, and the experimental law asserts that,

$$\frac{v' - u'}{u - v} = e.$$

Two inelastic balls impinging directly.

37. *Two inelastic balls impinge directly with given velocities, to find their velocity after impact.*

Let A, B be the masses of the two balls, and u, v their velocities estimated in the same direction; then, after impact, they proceed with a common velocity, V (suppose). The algebraic sum of the momenta before impact is

$$Au + Bv;$$

and, after impact, it is

$$(A + B)\ V.$$

Hence, by the law of equality of momenta,

$$(A + B)\ V = Au + Bv, \text{ and}$$

$$V = \frac{Au + Bv}{A + B}.$$

If the second ball were moving in an opposite direction to the first, v would be taken negative, and the direction of V will be indicated by its resulting sign.

Cor. 1. If B were at rest, then $v = 0$, and

$$V = \frac{A}{A + B}u.$$

Cor. 2. If the balls be brought to rest by the impact, then $V = 0$, and, therefore,

$$Au + Bv = 0,$$

$$\text{or } \frac{u}{-v} = \frac{B}{A}.$$

Or the balls must have been moving in opposite directions with velocities inversely proportional to their respective masses.

38. *Two elastic balls impinge directly with given velocities, to determine their velocities after impact.* Two elastic balls impinging directly.

Let A, B, be the masses of the two balls:

u, v, their velocities before impact, estimated in same direction,
u', v', " after " " "
e, the elasticity.

A is supposed to overtake B.

Then the sum of their momenta before impact is $Au + Bv$.
and " " after " $Au' + Bv'$.

Hence, by the law of equality of momenta,

$$Au' + Bv' = Au + Bv.$$

Again, by the law of relative velocities,

$$v' - u' = e(u - v).$$

From these two equations, finding u' and v', we have

$$u' = \frac{Au + Bv - Be(u - v)}{A + B},$$

$$v' = \frac{Au + Bv + Ae(u - v)}{A + B}.$$

If B were moving before impact in an opposite direction, v would be taken negative, and the directions of u', v' will be indicated by their algebraic signs.

Cor. 1. In no case can both balls be brought to rest by the impact.

Cor. 2. If the balls be perfectly elastic, or $e = 1$; and if also their masses be equal, or $A = B$; then we have Two equal and perfectly elastic balls exchange velocities

$$u' = v, \; v' = u,$$

and the two balls exchange velocities.

Thus, if the second were at rest, the first after impact would remain at rest, and the second would go on with the velocity of the first before impact.

Cor. 3. Hence, if a row of equal, perfectly elastic balls be ranged in contact in a straight line, and another ball, also perfectly elastic and equal to each of them, impinge in that line on the first of these balls, the impinging ball will remain at rest after the impact, and the first ball will start with the same velocity; it will then impinge on the second, communicating to it this velocity, and itself remaining at rest; the second on the third, and so on, till the last ball flies off with the velocity, all the others remaining at rest.

Again, if two such balls, moving with the same velocity, impinge on the row of balls, the first of the impinging balls will, as before, drive off the last of the row; and the second will then drive off the last but one, all the others remaining at rest. And so on for any number of impinging balls, whether greater or less than that of the balls struck, the number of balls driven off will be the same as that of the impinging balls, the others remaining at rest.

Impact on a fixed body, deduced.

Cor. 4. In the general case, suppose B to be at rest before impact, then $v = 0$, and we have

$$u' = \frac{A - Be}{A + B} u, \qquad v' = \frac{A + Ae}{A + B} u.$$

Now suppose B to become indefinitely great compared with A; then the limiting values of $\frac{A - Be}{A + B}$ and $\frac{A + Ae}{A + B}$ are $-e$ and 0. Hence,

$$u' = -eu, \; v' = 0,$$

and we have the case of a ball impinging directly on a *fixed* body, and the first equation shows that the velocity after impact is in the opposite direction to that before impact, and its magnitude is less in the ratio of e to 1.

39. *Oblique impact of smooth balls.*

Oblique impact.

Here the centres of the balls are not moving in the same line, but the impulsive action takes place (the balls being smooth) in the line joining their centres at impact. If therefore the velocity of each ball be resolved in two directions; one, along the line joining the centres; the other, perpendicular to this line; the latter resolved parts will not be altered by the impact, and the former will be altered precisely as if the

impact were direct, and the resolved velocities in this direction after impact can be calculated by the preceding investigation, and then compounded with those in the perpendicular direction by the parallelogram of velocities, thus determining fully the motion and direction of motion of each ball.

40. *Oblique impact against a smooth fixed plane.* Impact at a fixed plane.

The impulsive action exerted by the plane is in a normal direction, the plane being smooth. Hence, if we resolve the velocity of the ball in two directions; one, along the plane; the other, normal to it: the former will be unaffected by the impact, while the latter will be changed just as if the impact were direct, that is, its direction will be reversed and its magnitude diminished in the ratio of $e:1$. Combining these velocities, the motion and direction of motion of the ball after impact are determined. Thus:

Let v be the velocity of the particle, θ the angle its direction makes with the normal to the plane at the point of impact.

Let v' be the velocity after impact, and θ' the corresponding angle. Then $v\cos\theta$, $v\sin\theta$ are the velocities respectively normal to and along the plane, and we have

$$ev\cos\theta = v'\cos\theta',$$

$$v\sin\theta = v'\sin\theta';$$

from which we obtain

$$v' = v\sqrt{\left\{\sin^2\theta + e^2\cos^2\theta\right\}}, \text{ and}$$

$$\tan\theta' = \frac{1}{e}\tan\theta.$$

Or, the same may be done by an easy geometrical construction; thus, take AC to represent the velocity at impact, CM the normal, AM perpendicular to CM. Then AM, MC represent the component velocities along and perpendicular to the plane. Take $CN = e\,CM$, NB perpendicular to CM and equal to MA; join CB: then CB will represent in magnitude and direction the velocity after impact. Geometrical construction. Fig. 6.

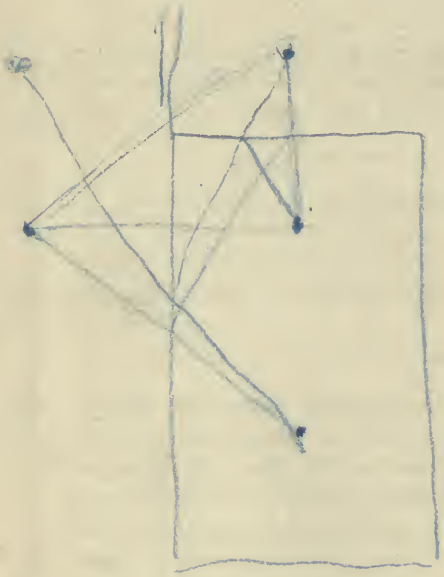
With perfe elasticity, the angles o incidence and reflexion are equal.

Cor. If the elasticity be perfect, the velocity is the same after impact as before, and its direction is equally inclined to the normal on the other side of it.

This furnishes a simple construction for finding the direction in which a particle in a given position must be projected so as after reflexion at a fixed plane (perfectly elastic) to strike a given point. The rule will be: aim at a point directly behind the plane at the same distance from it as the point required to be hit.

How to hit a required point after successive reflexions a any numbe of planes.

So also, if it be desired to strike a given point after reflexion at two fixed planes in succession, imagine a point at the same distance directly behind the second plane as the given point is before it, and then aim at a point directly behind the first plane at the same distance from it as this imaginary point is before it. And a similar construction serves for the same problem after successive reflexions at any number of planes.



APPENDIX.

A COLLECTION OF PROBLEMS;

ADAPTED TO

CHERRIMAN'S MECHANICS;

CHIEFLY SELECTED FROM THE EXAMINATION PAPERS OF THE UNIVERSITY OF TORONTO.

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APPENDIX.

CHAPTERS I. & II.

A perfectly smooth surface is one capable of exerting a pressure only in a direction at right angles to itself. In representing the unknown reactions which smooth surfaces exert on bodies in contact with them, this definition or fact furnishes us at once with an important element in such reactions—direction.

The tension of a flexible string is the same throughout. Hence if a string passes over a smooth peg or through a smooth ring, the resultant pressure on the peg or ring bisects the angle between the parts of the string. Important applications of this fact will also be found when the mechanical powers come to be treated of.

It will frequently assist in the solution of problems to remember that if three forces acting on any body maintain equilibrium, their directions must lie in the same plane, and either pass through the same point or be parallel.

It should also be noted that when three forces acting at a point keep it at rest, their directions may be turned in their plane through equal angles in the same sense without disturbing the equilibrium; hence it follows that three forces will keep a point at rest when their magnitudes are represented by the sides of a triangle, and their directions are equally inclined to the respective sides towards the same parts.

---

1. If a force which can just sustain a weight of 5 lbs., be represented by a straight line whose length is 1 ft. 3 in., what force will be represented by a straight line 2 ft. long?

2. How would a force of a ton be represented, if a straight line an inch long were the representation of a force of 50 lbs.?

3. If a force of P lbs. be represented by a straight line a inches long, by what straight line will a force of Q lbs. be represented?

4. If a force of P lbs. be represented by a straight line a inches long, what force will be represented by a straight line b inches long?

5. If a certain distance measured off on a straight line represent a certain force when 2 inches and 3 ounces are the units of length and force, compare this distance with the distance which represents the force when 3 inches and 6 ounces are the units of length and force.

6. If a line of length p represent a force, what will represent it when the units of length and force are changed in the ratios of $a : b$ and $c : d$ respectively?

7. State the experimental law on which the science of statics is based.

If a rod be pressed against a wall in the direction of its length, are you to infer from the above law that the actual pressure thus caused at any point in the wall, in the direction of the rod produced, is equal to that at the point of contact of rod and wall? Explain clearly the meaning of the law.

---

1. What principles are assumed in the demonstration of the parallelogram of forces?

2. Shew that the line drawn from an angle of a triangle to the middle point of the opposite side, will represent one-half of the resultant of the forces which are represented by the sides of the triangle terminated in that angular point.

3. If three forces acting on a rigidly connected system of particles maintain equilibrium, their directions must lie in the same plane, and either pass through the same point or be parallel.

4. Two forces of 3 and 4 lbs. act at right angles to one another; find their resultant.

5. Two forces, one of which is double the other, act upon a particle in directions making with each other an angle of 120°; find their resultant.

6. A particle is kept at rest by three forces acting at a point; the angle between the directions of two of the forces whose magnitudes are 1 and 2 respectively, is 120°; find the magnitude and direction of the third force.

7. Three forces acting at a point keep it at rest; two of them are 4 and $4\sqrt{3}$ respectively, and they make an angle of 150° with one another; find the other force and the angles it makes with these.

8. Two forces, one of which is double the other, act upon a particle in directions inclined to each other at an angle of 120°; shew that the direction of the resultant is perpendicular to the direction of the smaller force.

9. If two forces of 2 and 3 lbs. have for resultant $\sqrt{7}$ lbs., find angle at which they act.

10. If two forces and their resultant be respectively $\sqrt{5}+\sqrt{3}$, $\sqrt{5}-\sqrt{3}$, and $2\sqrt{5}$, find the angles at which the forces are inclined to one another.

11. Two forces are in the ratio of 3 : 2; find the angle between their directions when the resultant is a mean proportional between them.

12. If two forces acting at right angles to each other be in the proportion of 1 to $\sqrt{3}$, and their resultant be 10 lbs., find the forces.

13. A force $3\sqrt{3}$ has a component 6 in one direction; find the other component, the direction of the components making an angle 120° with one another.

14. Three forces acting in the same plane keep a point at rest; the angles between their directions are 135°, 120° and 105°; compare their magnitudes.

15. Three forces acting at a point are in equilibrium when they make angles of 60°, 135°, 165° with each other; find ratio of the forces.

16. Three forces represented by the numbers 3, 5 and 9, cannot under any circumstances produce equilibrium upon a point.

17. Three forces acting at a point keep it at rest: if one of them be reversed, find the direction and magnitude of the resultant.

18. In the parallelogram of forces, if the two sides of the parallelogram approach coincidence in the same or opposite directions, what is the magnitude of the resultant ultimately?

19. Resolve a given force into two others at right angles to each other, one of which shall be half the original force.

20. Two equal forces sustain each other by means of a string passing over a smooth tack; prove that either force : pressure on tack : : $\frac{1}{2}$: cosine of half the angle between the directions of the forces.

21. Three smooth pegs form a vertical equilateral triangle whose base is horizontal, and two equal weights (w, w) are connected by a string which passes over them; find pressure on each peg.

22. Shew that a force R can always be resolved into two, $R \sec \theta$, making an angle θ with, and $R \tan \theta$, perpendicular to the direction of R. What is the whole effect of R in the latter direction?

23. Shew how to find geometrically three pressures, which, acting at a point in given directions, will be equivalent to one given pressure acting at the same point.

24. Three forces represented by the numbers 1, 2, 3, act on a particle in directions parallel to the sides of an equilateral triangle taken in order; determine their resultant.

25. Forces of 2, 3, 4 and 5 lbs. act on a point in direction of the sides of a square taken in order: find their resultant.

26. A particle placed at the centre of an octagon, is acted on by forces in directions tending to each of the angles of the figure, and of magnitudes, taken in order, 2, 4, 6, 8, 10, 12, 14, 16; find the magnitude and direction of their resultant.

27. Two equal forces act at a point; through what angle must one of them be turned in order that their resultant may be turned through a right angle?

28. A segment of a circle being given, if two equal forces act along the sides of any angle of the segment, there is a point in the circumference of the circle, about which, if fixed, there will be equilibrium, whichever angle be taken, and whatever be the magnitudes of the forces.

29. Three forces acting on a particle in one plane keep it at rest: of the six elements, viz., the magnitudes of the forces and the angles they make with one another, shew that in general three at least must be given to determine the rest. If the given parts are the angles, what is the only inference?

30. If three forces act at a point, and be represented by three lines drawn from that point such that each of them would, if produced backwards, bisect the distance between the extremities of the other two, the point will be kept at rest.

31. A, B, C and D are fixed points, and P, a moving point, is always acted on by forces represented in magnitude and direction by PA, PB, PC, PD; if PQ always represent the resultant force on P, shew that PQ always passes through a fixed point O, such that the ratio of PO to OQ is constant.

32. Assuming the truth of the parallelogram of forces for the magnitude of the resultant, prove it for the direction of the resultant.

33. Given R, P, α, in the formula $R^2 = P^2 + Q^2 + 2PQ \cos \alpha$, shew in what cases there will be two different values of Q which satisfy the problem; and if these conditions be fulfilled, show that the arithmetic mean between the two values of Q is equal to the rectangular component of P in the direction opposite to that of Q.

34. If two forces acting at a point be represented in magnitude and direction by the diagonals of a quadrilateral, their resultant will coincide with that of the forces represented by two opposite sides.

35. $ABCD$ is a parallelogram; find the resultant of the forces represented by AD and AB acting at A, and BA and AC acting at B.

36. ABC is a triangle; in BC take a point D such that $BD = \frac{1}{3} BC$. Shew that $3AD$ will represent the resultant of forces represented by AB, $2AC$ and CB respectively, all the forces being supposed to act at A.

37. ABC is a triangle; in CB take a point D such that CD is $\frac{1}{4}$th of CB. Shew that $4AD$ will represent the resultant of forces represented by BC, $2AB$ and $2AC$ respectively; all the forces being supposed to act at A.

38. D is a point within a circle, and A any point on its circumference; determine two chords, AB, AC, which, if taken to represent in magnitude and direction two forces acting at A, will have AD for their resultant.

39. In a triangle obtain a geometrical construction for the point from which, if strings sustaining weights proportional to the opposite sides pass over pulleys at the angular points of the triangular, the point will remain at rest.

40. O is the centre of a circle about a triangle, on sides of which OD, OE, OF are perpendiculars: forces act along OD, OE, OF proportional to them in magnitude. Shew that if O be kept at rest the triangle must be equilateral.

41. If $2n$ forces acting on a point be represented in magnitude and direction by pairs of lines drawn from a point (not the centre) within a circle to the circumference in opposite directions, prove that there must always be a resultant whose inclination to the diameter passing through the point is $\tan^{-1}\frac{\Sigma(\sin 2\theta)}{n+\Sigma(\cos 2\theta)}$, θ being the angle between one of the forces and the diameter through the point.

42. If four forces act at a point and are represented in magnitude and direction by the sides of a quadrilateral, either they keep the point at rest, or have a resultant which is double a force represented by one of the sides or one of the diagonals.

43. If six sides of a regular octagon taken in order represent in direction and magnitude the forces acting on a point, find two equal forces which, acting at right angles to one another, will keep the point at rest.

CHAPTER III.

Since twice the area of a triangle is the product of the base by the perpendicular height, it follows that the moment of a force about a point may be represented by twice the area of the triangle whose base is the line representing the force, and vertex this point. Hence also, if three forces acting on a body are represented in magnitude and position by the sides of a triangle taken in order, their resultant moment about any point in the plane of the triangle is represented by the double of its area. A similar proposition holds when the forces are represented by the sides of a plane polygon, provided the sides do not cross.

Forces acting in one plane on a particle will be in equilibrium if the algebraic sum of their moments vanish about two points in the plane which are not situated on a straight line through the particle. For let A be the particle, and B, C, two such points. Then since the sum of the moments vanishes about B, if there be a resultant, it must pass through B. Similarly it must pass through C. But it passes through A. Therefore it must pass through A, B, and C; which is impossible, since they are not in the same straight line. Hence the resultant vanishes. In problem 4 will be found an application of this principle.

In general, when forces acting in one plane maintain equilibrium, by resolving them in two directions at right angles to each other, and equating to zero the algebraic sums of the separate sets of components, we obtain two equations—relations between the quantities, known and unknown, involved in the problem. A third relation is obtained by equating to zero the algebraic sum of their moments about any point in their plane. If the data be sufficient for the complete solution of the problem, and more than three unknowns be involved, additional equations

must be obtained from considering the geometrical relations of parts of the system. It must be noted that if the two equations above spoken of are formed, and a third by taking moments about any point, no fresh independent relations are furnished by resolving the forces in other directions, or by taking moments about any other point.

It will at once be seen that the statement in *Italics* at the end of § 35 applies to the case of any number of parallel forces in one plane. The magnitude of their resultant is found by taking their algebraic sum, and its position from the fact that its moment about any point in their plane is equal to the algebraic sum of their moments.

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1. If a force be represented by a straight line, what is the geometrical representation of the moment? If an inch be taken as the unit of length, what is the geometrical representation of the unit of moment? When is a force said to have a negative moment?

2. Three forces are represented in magnitude and position by the sides of a triangle taken in order; shew that the moment of the forces about any point in the plane of the triangle is represented by twice the area of the triangle.

3. If the algebraic sum of the moments of forces acting in one plane on a particle, has the same value for two points in the plane, then the straight line which joins these two points is parallel to the resultant force.

4. Forces act at the angular points of a triangle along the perpendiculars drawn from the angular points on the respectively opposite sides, each force being proportional to the side to which it is perpendicular. Shew, by taking moments round the angular points, that the forces are in equilibrium.

5. Shew that the proposition contained in § 34 must hold in the case of parallel forces without assuming the know-

ledge of the magnitude or position of the resultant in this case.

6. Three parallel forces acting on a rigid system keep it at rest; investigate the relations subsisting between their magnitudes, directions, and distances.

7. $ABCD$ is a square; a force of 3 lbs. acts from A to B, a force of 4 lbs. from B to C, and a force of 5 lbs. from C to D; if the centre of the square be fixed, find the force which, acting along AD, will keep the square in equilibrium.

8. A weight W is suspended freely from a fixed point by a perfectly flexible string; find what horizontal force applied to the string will draw the upper portion of it 30° out of the perpendicular.

9. A weightless rod AB, A being fixed and B sustaining a weight P, is supported by a string BC drawn to a point C vertically above A, AC being equal to AB, and the angle ABC 30°. Find the tension of the string.

10. If a cord, whose length is $2l$, be fastened at two points lying in the same horizontal line, at a distance $2a$ from each other, and if a smooth ring upon the cord sustain a weight W, prove that the tension of the cord is

$$\frac{W l}{2\sqrt{l^2 - a^2}}.$$

11. A weight P, suspended from one end of a lever without weight, is balanced by a weight of 1 lb. at the other end of the lever; when the fulcrum is removed through half the length of the lever, it requires 10 lbs. to balance P: find P.

12. Four weights, 1, 3, 7, 5, are placed at equal distances on a straight rod. Find their resultant and the point through which it acts.

13. The arms of a weightless lever are inclined to each other: shew that the lever will be in equilibrium with

equal weights suspended from its extremities, if the point midway between the extremities be vertically above or below the fixed point.

14. Two weights are suspended from the arms of a bent lever without weight, the arms being 10 ft. and 6 ft. long, and inclined to the horizon at angles 30° and 45° respectively. Find the ratio of the weights.

15. Two weights, P and Q, are suspended, one at each end of a bent lever, whose arms (a, b,) are inclined to each other at an angle α. If φ be the angle at which the arm a is inclined to the horizon when the lever is in equilibrium, prove

$$\tan \varphi = \frac{a\,P}{b\,Q} \operatorname{cosec} \alpha + \cot \alpha.$$

16. A bent weightless rod ABC, containing a given angle at B, and having suspended at A and C given weights P and Q, is fixed at B. Find the position of equilibrium.

17. In the preceding question, if Q be attached to B, and the whole suspended from C, find the position of equilibrium.

18. At what point of a tree must a rope of given length be fixed, so that a man endeavoring to pull down the tree may have the greatest advantage?

19. Two equal weights (W, W) are supported by a string which passes over three tacks, forming a vertical isosceles triangle, of which the base is horizontal, and the angle opposite to the base 120°. Find the pressure on each of the tacks.

20. Two uniform heavy rods, similar in every respect, are capable of motion in a vertical plane round a common fixed pivot at one extremity of each, and they are kept in equilibrium in a position inclined to the horizon by a string placed over the other ends and kept stretched by two equal weights at its extremities. Find the position of equilibrium.

(The entire weight of a uniform rod may be supposed collected at its centre. Chap. IV.)

21. A uniform heavy rod rests horizontally with its ends on two props; find at what point a weight equal to the weight of the rod must be hung, in order that the pressures on the props may be in the ratio 2 : 1.

22. A bent lever consists of two uniform heavy beams whose lengths are as 1 : 2; find the weight which must be attached to the extremity of the shorter in order that the arms may make equal angles with the horizon.

23. A uniform heavy rod AB, placed in a smooth circle whose plane is vertical, will be kept in equilibrium in any position by a weight applied at A proportional to $\frac{BC}{AC}$, AC being a horizontal chord.

24. One end of a heavy uniform rod rests against a smooth vertical wall. A smooth ring without weight, attached to a point in the wall by an inextensible string, slides on the rod. If θ be the angle which the rod makes with the wall when in equilibrium, and if the length of the rod be $2m$ times that of the string,

$$\cot^3\theta + \cot\theta - m = 0.$$

25. AB is the diameter of a circle movable about A. BC, BD are chords on opposite sides of BA : along and proportional to BC, BD two forces are acting. Shew that if BC is fixed in the circle, BD may in general occupy two different positions, and equilibrium be maintained.

26. P and Q are fixed points on the circumference of a circle; QA and QB are any two chords at right angles to each other, on opposite sides of QP: if QA, QB denote forces, shew that the difference of their moments about P is constant.

27. Five forces are represented in magnitude and position by five of the sides of a regular hexagon taken in order; shew that the moment of the resultant of the forces about

the intersection of the sides of the hexagon adjacent to the side in which no force acts is represented by $\frac{7}{2} a^3 \sqrt{3}$, a side of the hexagon being equal to a.

28. Ten men of equal strength, wishing to pull down a tree of height h, fix two ropes to its top, one of which reaches the ground, making an angle of 60° with the tree, and the other 45°. If four men pull at the former and six at the latter, and in directions along the ground making an angle of 45° with one another, find the moment of the force which is effective in pulling down the tree.

29. AB is a given horizontal line, BC a weightless rod moving freely in a vertical plane about B. A weight is suspended by a string fixed at A and passing over the end C of the rod. Find the pressure at B.

30. A triangular lamina ABC fixed at the centre of its circumscribing circle is in equilibrium, (1) when the forces P, Q, R, act along its sides a, b, c, respectively; (2) when the same forces act in directions perpendicular to the former ones at the points B, C, A, respectively; prove

$$\frac{P}{\sin (B-C)} = \frac{Q}{\sin (C-A)} = \frac{R}{\sin (A-B)}.$$

What follows from these relations if two or all the angles are equal?

CHAPTER IV.

1. ABC is a triangle; parallel forces Q and R act at B and C, such that Q is to R as tan B is to tan C; shew that their centre is at the foot of the perpendicular from A on BC.

2. Parallel forces P, Q, R act at the angular points A, B, C of a triangle; shew that the perpendicular distance of their centre from the side BC is

$$\frac{P}{P+Q+R} \times \frac{2 \text{ area of triangle}}{BC}.$$

3. Parallel forces P, Q, R act at the angular points A, B, C of a triangle, and their centre is at O; shew that

$$\frac{P}{\text{area of } BOC} = \frac{Q}{\text{area of } COA} = \frac{R}{\text{area of } AOB}.$$

4. Equal like parallel forces act at five of the angular points of a regular hexagon; determine the centre of parallel forces.

5. The circumference of a circle is divided into n equal parts, and equal like parallel forces act at all of the points of division except one; find the centre of parallel forces.

6. Two parallel forces being in the ratio of $\sqrt{2}+1$ to 1, shew that if one of them be reversed, the resultant will be moved through a distance equal to that between the two forces.

7. If of three parallel forces acting in the same sense the middle one is the resultant of the other two, prove that in whatever senses the forces may be severally acting, the line of action of the resultant of the three coincides with that of one or other of the forces.

8. A rigid parallelogram can turn about a fixed point in one of its diagonals, and is acted on by four forces along its sides (which also represent them respectively in magnitude), and each pair of parallel forces tend towards the same parts. Prove that either the parallelogram is at rest, or it can be kept so by a force whose moment round the fixed point is represented by four times the area of the triangle formed by joining the fixed point to the ends of that diagonal in which it is not situated.

9. A uniform rod 3 ft. long and weighing 4 lbs. has a weight of 2 lbs. placed at one end; find the centre of gravity of the system.

10. Two weights of 5 and 7 lbs. are connected by a rod, without weight, measuring 6 ft.; find their centre of gravity, and determine the change which will take place in its position if 1 lb. be added to the smaller weight.

11. A uniform beam, 18 ft. long, rests in equilibrium upon a fulcrum 2 ft. from one end, having a weight of 5 lbs. at the end furthest from the fulcrum, and one of 110 lbs. at the other. Find the weight of the beam.

12. A beam 30 ft. long, balances about a point at one-third of its length from the thicker end; but when a weight of 10 lbs. is suspended from the smaller end, the fulcrum must be moved 2 ft. towards it in order to maintain equilibrium. Find the weight of the beam.

13. What particular case of the *centre of parallel forces* constitutes the *centre of gravity*?

Find the centre of gravity of a number of heavy particles lying in a straight line, and shew that if the weight of each be increased by the same quantity, their centre of gravity will be shifted to the right or left according as it lay to the left or right of the centre of gravity of a number of particles occupying the same places, but all having the same weight.

14. Four particles whose weights are as 1, 2, 3, 4, are placed in order at corners of a square; shew that their centre of gravity lies on a line through the centre of the square parallel to two sides, and at a distance from the centre equal to one-fifth of side of square.

15. Find the centre of gravity of four equal particles placed at the corners of a quadrilateral figure.

16. A rod AB, capable of turning about the end A, rests with the end B against a smooth vertical wall; construct for the direction of the reaction at A, and shew that it bears to the weight of the rod the ratio of $\sqrt{5}$ to 2, the rod making an angle of 45° with the vertical.

17. What propositions are necessary to shew that every body has one, and only one, centre of gravity?

18. Prove that every system of bodies possesses one, and only one, centre of gravity.

19. Define stable and unstable equilibrium, and explain why an egg, when resting on its side, if canted towards an end, will return to its original position, but if, when on its end, it be canted towards a side, will not.

20. Three equal particles are placed at the middle points of the sides of a given triangle; find their centre of gravity.

21. At the middle points of the sides of a triangle are placed three equal weights; prove that the sum of the squares of their distances from their centre of gravity is $\frac{1}{12}(a^2 + b^2 + c^2)$, where a, b, c are the sides of the triangle.

22. Three weights of 1, 2, and 3 lbs. are placed at the angles of an equilateral triangle; determine the distance of their centre of gravity from each of the angles of the triangle in terms of the length of a side.

23. If there be four particles A, B, C, D, and E, the intersection of AC and BD, be their centre of gravity, then E is also the centre of gravity of A and C, and of B and D.

24. The centre of gravity of four equal particles in any position is the same as that of four other equal particles, each of which is placed at the centre of gravity of three of the former.

25. $4n$ particles of equal weight are situated at the ends of chords of a circle, which pass through the same point and are, two and two, at right angles to one another. Shew that the centre of gravity of the system is the bisection of the line from the centre to the point through which the chords pass.

26. If forces act at the centre of gravity of a system of equal heavy particles, represented in magnitude and direction by the straight lines drawn from that point to each particle of the system, they will be in equilibrium.

27. The sum of the products of the weights of any number of particles by the squares of their respective distances from a given point, is less when that point is the centre of gravity of the system than when it is any other.

28. If two triangles be on the same base, shew that the straight line which joins their centres of gravity is parallel to the straight line which joins their vertices.

29. G is the centre of gravity of a triangle ABC, O any point in its plane. The forces represented by OA, OB, OC, acting at O, have for resultant the force represented by $3OG$.

30. Shew that the centre of gravity of a plane triangular area coincides with that of three equal weights placed at its angular points.

In what case will the centre of gravity of a quadrilateral coincide with that of four equal weights placed at its angular points?

31. If the centre of gravity of three heavy particles placed at the angular points of a triangle coincides with the centre of gravity of the triangle, the particles must be of equal weight.

32. If the centre of gravity of a triangle be equidistant from two angular points of the triangle, the triangle must be isosceles.

33. If three men support a heavy triangular board at its three corners, compare the forces exerted by each man.

34. The locus of the centres of gravity of all right-angled triangles on the same hypothenuse (a) is a circle of radius $\frac{a}{6}$.

35. If G be the centre of gravity of the triangle ABC, then

$$3\,(GA^2 + GB^2 + GC^2) = AB^2 + BC^2 + CA^2.$$

36. If the sides of a triangle are 3, 4 and 5, determine the distance of its centre of gravity from either of the angles.

37. Determine the length of a straight line drawn through the centre of gravity of a given isosceles triangle, making a given angle with the base, and terminated by the sides.

38. Find geometrically the centre of gravity of a plane quadrilateral lamina.

39. $ABCD$ is a parallelogram having the angle $ABC = 60°$, and the base BC of given length; determine the greatest possible length of AB if the figure is to stand on BC.

40. A quadrilateral lamina is suspended by each of the middle points of its sides in succession, and the intersection of the diagonals in each case lies vertically under the point of suspension; prove that the quadrilateral is a parallelogram.

41. Find by a geometrical construction the centre of gravity of the four-sided figure formed by drawing a line parallel to the base of an isosceles triangle.

42. The diagonals AC, BD, of a quadrilateral meet in E, and AF is taken on AC equal to CE; shew that the centre of gravity of the quadrilateral lies on the line joining F to the middle point of BD.

43. Prove the following construction for finding the centre of gravity of a quadrilateral $ABCD$. Take E the middle point of CD, join EA, EB, and in them take F, G, points such that $EF = \frac{1}{3} EA$, $EG = \frac{1}{3} EB$, and through FG draw lines parallel to DB, AC respectively, intersecting in H; this point will be the centre of gravity.

44. EBF is a triangular section of the uniform quadrilateral lamina $ABCD$, weighing $\frac{1}{n}$th of the whole; if EBF be transferred along EF till its centre of gravity is in the line AB or BC, the centre of gravity of the whole figure will be shifted parallel to EF through the distance $\frac{EF}{3n}$.

45. A plane quadrilateral $ABCD$ is bisected by the diagonal AC, and the other diagonal divides AC into two parts in the ratio of p to q; shew that the centre of gravity of the quadrilateral lies in AC, and divides it into two parts in the ratio of $2p+q$ to $p+2q$.

46. Find the centre of gravity of three plane laminæ of equal thickness, joined so as to form a pyramid, any section of which, by a plane parallel to the base, is a right-angled triangle.

47. A quarter of a triangle is cut off by a straight line drawn parallel to one of the sides; find the centre of gravity of the remaining piece.

48. A triangle ABC has the sides AB, BC equal; a portion APC is removed such that AP and PC are equal; compare the distances of B and P from AC in order that the centre of gravity of the remainder may be at P.

49. Find the centre of gravity of a uniform circular disc out of which another circular disc has been cut, the latter being described on a radius of the former as diameter.

50. The middle points of two adjacent sides of a square are joined, and the triangle formed by this straight line and the edges cut off; find the centre of gravity of the remainder of the square.

51. $ABCD$ is a square, AC one of its diagonals; out of the triangle ACD a square (one of whose sides coincides with AC) is cut; shew that the distance of the centre of gravity of the remainder from $B = \frac{8a}{7\sqrt{2}}$, a being a side of the square $ABCD$.

52. Out of a square it is required to cut a triangle having one side for its base, such that the centre of gravity of the remainder shall be in its vertex.

53. A prism on a triangular base has a small orifice drilled through its centre of gravity parallel to its edges. Prove that, if the prism be cut in any manner by a plane, the section of the orifice will coincide with the centre of gravity of the triangular section of the prism made by the plane.

54. The centre of gravity of a triangle divides in the ratio of 2 : 1, the line which joins the intersection of perpendiculars from the angles on the opposite sides, and the centre of the circumscribing circle.

55. CA and CB are the arms of a uniform bent lever; determine the distance of the centre of gravity from C in terms of the lengths of the arms and the angle ACB.

56. A heavy bar 14 ft. long is bent into a right angle so that the lengths of the portions which meet at the angle are 6 and 8 ft. respectively; shew that the distance of the centre of gravity of the bar so bent from the point which was the centre of gravity when the bar was straight, is $\frac{9\sqrt{2}}{7}$ ft.

57. If the sides of a triangle ABC be bisected in the points D, E, F, the centre of the circle inscribed in the triangle DEF will be the centre of gravity of the perimeter of the triangle ABC.

58. Shew that the distances of the centre of gravity of the perimeter of a triangle from the sides a, b, c, are as

$$\frac{b+c}{a} : \frac{c+a}{b} : \frac{a+b}{c}.$$

59. Prove by statical considerations (or otherwise) that the lines drawn through the middle points of the sides of a triangle parallel to the bisectors of the opposite angles meet in a point. If P be this point for the triangle ABC, and O be the centre of the inscribed circle, the three forces represented by OA, OB, OC will have $2\,OP$ for resultant.

60. A heavy uniform wire is bent into the form of an equilateral triangle, which is loaded at the angular points with weights in the proportion of $3:4:5$; find the centre of gravity of the system, the sum of the weights being equal to the weight of the wire.

61. A thin uniform wire is bent into the form of a triangle ABC, and heavy particles of weights P, Q, R are placed at the angular points; prove that, if the centre of gravity of the weights coincide with that of the wire,

$$P : Q : R :: b+c : c+a : a+b.$$

62. A right-angled isosceles triangle ABC, C being the right angle, is fixed at A, and BC is kept horizontal by a horizontal force P applied at B. If W be the weight of the triangle, shew that $P = \frac{1}{3}\,W$. Find also the direction of the resultant pressure at A.

63. A triangular lamina ABC at rest with the side BC downwards, is supported by a rod through the centre of the inscribed circle; shew that if BC be the greatest side of the triangle, the equilibrium will be stable; if least, unstable.

64. A uniform wire is formed into a triangle ABC, right-angled at C, and is suspended from C; θ is the angle which the side AC makes with the vertical; shew that $\tan\theta = \frac{a}{b} \cdot \frac{a+c}{b+c}$.

65. Three uniform rods form a triangle ABC, which is suspended at the middle point of BC with the vertex downwards; shew that the inclination of BC to the horizon is half the difference between the angles at B and C.

66. Three uniform rods of different materials are rigidly connected in the shape of a triangle, and suspended from one angle, the weights of the sides respectively opposite and adjacent to this angle being as 3 to 2 to 1. Shew that in the position of rest, the perpendiculars from the other angles upon the vertical through the point of suspension are as 5 to 4.

67. A lamina of the form of an equilateral triangle, is hung up against a smooth vertical wall by means of a string attached to the middle point of one side, so as to have an end of this side which is uppermost, in contact with the wall; determine the position of equilibrium.

68. ABC is a uniform triangular lamina in a vertical plane, right-angled at C, and capable of moving freely about an axis through C perpendicular to its plane. A weight w is attached at B; if $3w$ be the weight of the triangle, shew that it can be kept at rest, with the base BC horizontal, by a force $2w$ cosec B acting at any point in BA in direction of its length.

69. A triangular lamina is poised on its centre of gravity; shew that the balance will not be destroyed by circumscribing the lamina with a triangle formed by three straight wires of equal weight, the middle points of the wires being fastened to the angular points of the lamina.

70. A right-angled triangle is hung from a smooth peg by means of a string fastened to the ends of the hypothenuse

BC, and it is found that one or the other side is vertical according as the length of the string is l_1 or l_2; prove that

$$l_1^2 + l_2^2 = 10\ BC^2.$$

71. A heavy rod is movable freely about a fixed point in itself, and to one end of it is attached a string bearing a given weight, while to the other end is attached another string of given length carrying a smooth ring through which the former string passes. Find equations for determining the positions in which the rod will rest.

72. Particles of equal weight are placed at the n angular points of a polygon. Each particle is in turn omitted, and the centre of gravity of the remaining particles found; shew that if the points thus found successively be joined, a polygon similar to the original will be formed, and that the homologous sides will be in the ratio of $1 : n - 1$.

73. Shew how to place three unequal heavy particles on the circumference of a circle, so that their centre of gravity may coincide with the centre of the circle.

74. Perpendiculars are drawn from the angular points of a given triangle ABC upon the opposite sides, and another triangle is formed by joining the feet of these perpendiculars; prove that, if p, q, r be the distances of the centre of gravity of this triangle from the sides opposite to A, B and C,

$$\frac{p}{a^2 \cos(B-C)} = \frac{q}{b^2 \cos(C-A)} = \frac{r}{c^2 \cos(A-B)}.$$

75. Four equal particles are placed at the centres of the inscribed and escribed circles of a triangle; shew that their centre of gravity is at the centre of the circumscribing circle.

CHAPTER V.

Levers.

1. Two weights, 3 and 4, balance on the extremities of a lever 4 ft. long; find the fulcrum.

2. In an ordinary lever of the first class the forces P and Q are inclined inwards, each at an angle of 45° to the lever. Shew that the resultant pressure on the fulcrum is $\sqrt{P^2 + Q^2}$.

3. AC, CB are the equal arms of a straight lever whose fulcrum is C; to C a heavy arm CD is fixed perpendicular to AB. Prove that if a weight be suspended to the extremity A, and the system be in equilibrium, the tangent of the inclination of CD to the vertical will be proportional to the weight.

4. A straight lever of uniform thickness, the length and weight of which are given, has two weights, P and Q, attached to its extremities, and is sustained partly by a fulcrum at a given point, and partly by a peg, the pressure on which is known: required the position of the peg.

5. If a uniform beam of length a and weight W be placed with one end on a fulcrum, and a weight $2W$ be placed at the other end, shew that, if a power P be applied perpendicularly to support the whole, the beam must be horizontal, and the distance of the point of application of P from the end on which is placed the weight cannot be more than $\frac{a}{6}$.

6. There are n weights, W_1, W_2 ..... W_n in geometrical progression, and W_1, placed at A, one extremity of a lever, balances W_2, placed at B, the other extremity. Prove that a weight equal to the first $n-1$ weights, if placed at A, will balance a weight equal to the last $n-1$, if placed at B.

7. Two forces of 6 and $2\sqrt{3}$ lbs. act at ends of a lever, and are inclined to the direction of the lever produced at angles of 30° and 60° respectively. Determine the position of the fulcrum, and the *resultant* pressure on it.

8. The lever AC, without weight, turning about the fulcrum C, has two weights W, W', suspended from the extremity A and the middle point B respectively, and is kept at rest by the given weight P acting at A by means of a string passing over a tack at D; CD is horizontal and equal to AC; find the position of equilibrium.

9. A uniform bent lever, when supported at the angle, rests with the shorter arm horizontal; if the shorter arm were twice as long, it would rest with the other horizontal; compare the lengths of the arms, and find the angle between them.

10. A heavy uniform rod ABC is bent at C, and there rests on a fulcrum, its arms making angles α, β, with the vertical. If a weight equal to that of the shorter arm be attached to its extremity, these angles are reversed; prove $\sin \beta = \sqrt{3} \sin \alpha$, α being the angle the longer arm first makes with the vertical.

11. Investigate the conditions of equilibrium of a bent lever, acted on by two forces in any direction in the plane of rotation.

12. If a balance be false, having its arms in the ratio of 15 to 16, find how much per lb. a customer really pays for tea which is sold to him from the longer arm at 60 cents per lb.

13. A body, the weight of which is one lb., when placed in one scale of a false balance, appears to weigh 14 oz.; find its apparent weight when placed in the other scale, the inaccuracy of the balance arising from the unequal lengths of the arms. What would be the apparent weight, if the inaccuracy arose from the unequal weights of the pans, the arms being equal?

14. If one of the arms of a balance is twice as long as the other, and a weight of one pound is put in the scale attached to the shorter arm, will a customer gain or lose by buying two pounds, weighing one pound in each scale?

15. Explain why, in the common letter balance, it is not necessary to place the weights in any particular position on the pan.

16. If W_1, W_2 be the apparent weights of a body when placed successively at the two ends of a balance, what is its true weight (1) when the arms are unequal, (2) when one of the scales is loaded?

17. If a substance be weighed in a balance having unequal arms, and in one scale appear to weigh a lbs., and in the other b lbs., shew that its true weight is $\sqrt{ab}$ lbs.

18. In a false balance of which the arms are 10 and 10·1 inches, the weight used for the 1 lb. is really less than 1 lb. by the $\frac{1}{2020}$th part; shew that a person will get just weight by buying 2 lbs. of a substance, and having 1 lb. weighed separately in each scale.

19. A grocer, knowing that his balance has arms of unequal length, and wishing to act honestly, weighs (apparently) half of each of his sales in one scale, and half in the other; shew that he will not effect his object, and find his loss or gain per cent., if the lengths of the arms be as a to b.

20. If a piece be broken off from the longer arm of the Roman steelyard, how is the truth of the instrument affected?

21. In the common steelyard what change is made in the position of the point from which the graduations start, by increasing the movable weight?

What change in the position of this point is made by increasing uniformly throughout it the density of the material of the beam?

22. Having given the weight of a steelyard, 3 lbs., the movable weight, 4 lbs., the distances of the fulcrum and centre of gravity from the point of suspension of the scale, 3 and 5 inches respectively, graduate the instrument.

23. In the common steelyard, let A be the point from which the substance to be weighed (Q) is suspended, C the fulcrum, W the weight of the beam and hook, G the centre of gravity of these, N the point of suspension of the fixed weight, D the position of N when $Q = O$, then if $Q = W$, prove that the rectangle $DN.GC$ = rectangle $AC.CD$.

24. If the common steelyard be correctly constructed for a movable weight P, shew that it may be made a correctly constructed instrument for a movable weight nP by suspending at the centre of gravity of the steelyard a weight equal to $n - 1$ times the weight of the steelyard.

25. In the common steelyard, if the beam be uniform, and its weight $\frac{1}{m}$ of the movable weight, and the fulcrum be $\frac{1}{n}$ of the length of the beam from one end, shew that the greatest weight that can be weighed is $\frac{2m(n-1)+n-2}{2m}$ times the movable weight, the body to be weighed being attached to the end of the beam.

Pulleys.

1. If the string to which the weight is attached be coiled in the usual manner round the axle, but the string by which the power is applied be nailed to a point in the rim of the wheel, find the position of equilibrium, the power and weight being equal.

2. In the wheel and axle if straps be used, coiled each on itself, and their thickness be not neglected, find whether the ratio of the power to the weight would be increased or

diminished as the weight was raised, the straps being of the same thickness.

3. If there be two strings at right angles to each other, and a single movable pulley, find the force which will support a weight of $\sqrt{2}$ lbs.

4. Find the ratio of P to W in the single movable pulley, the strings making a given angle α with the horizon.

5. A man stands in a scale attached to a movable pulley, and a rope having one end fixed passes under the pulley and then over a fixed pulley; find with what force the man must hold down the free end in order to support himself, the strings being parallel.

6. In the first system of pulleys, if there be ten strings to the lower block, what power will support a weight of 1000 lbs.?

7. In the second system, if a man support a weight equal to his own, and there be three pulleys, find his pressure on the floor on which he stands.

8. In the second system of pulleys, draw a figure representing the angles the ropes make with each other when the free portion to which the power is applied is held in an inclined direction, the weight of the tackle being neglected.

9. In the second system of pulleys, shew that if the weight of each of the pulleys is equal to the power, no mechanical advantage is gained or lost by the system.

10. In a system of three pulleys, if a weight of 5 lbs. be attached to the lowest, 4 lbs. to the next, and 3 lbs. to the next, find the power required for equilibrium.

11. In the second system of pulleys, the number of the pulleys being five, and the weight of each being one ounce, find the weight of the counterpoise.

12. If there be three pulleys of equal weight, find the weight of each in order that a weight of 56 lbs., attached to the lowest pulley, may be supported by a power of 7 lbs. 14 oz.

13. In the second system of pulleys, if the weight of each pulley be w, shew that no mechanical advantage is gained unless the power P be greater than w. If W be the weight supported in this case, shew that by introducing another pulley, an additional weight $W - w$ will be supported.

14. Shew that 7 lbs. will support 49 lbs. on a system of three pulleys, each weighing 1 lb. What part of the power goes to support the 49 lbs., and what each of the pulleys?

15. In the second system, if no W act, and the weights of the pulleys be taken into consideration, shew that a force less than the weight of the heaviest pulley will sustain all of them.

16. In the third system of pulleys, there being 6 pulleys, what weight can be supported by a weight of 12 lbs.?

17. In the third system of pulleys, shew that a weight of three pounds will support twenty-five pounds on a system of two movable pulleys, each weighing one pound.

18. In the third system of pulleys, there being 8 equal heavy pulleys, find the ratio of the weight of one of the pulleys to the weight supported, in order that the latter may be supported by the weight of the pulleys alone.

19. A force of 6 lbs. supports a weight by means of 3 weightless pulleys arranged according to the third system. What force will be requisite to support the same weight by means of 8 pulleys arranged according to the first system, the weights of these pulleys forming an arithmetical progression, of which the first term is 1 oz., and the common difference $\frac{1}{2}$ oz.?

20. In the third system of pulleys, if the weight of each pulley be the same, and P be equal to the weight of all the movable pulleys (n in number), the mechanical advantage is

$$\frac{2}{n}(n+1)(2^n-1).$$

Inclined Plane.

1. What weight is that which it would require the same exertion to lift, as to draw a weight of 4 lbs. up a plane inclined at an angle of 30° to the horizon?

2. The power acting along the plane, if its length be 32 in. and height 8 in., find the mechanical advantage.

3. P acting along the plane, if $P=9$ lbs., find W when the height of the plane is 3 in. and the base 4 in.

4. An inclined plane rises 3 ft. 6 in. for every 5 ft. of length; if $W=200$, find P, supposing it to act along the plane.

5. What force is necessary to support a weight of 60 lbs. on a plane inclined at an angle of 60° to the horizon, the force acting horizontally?

6. The power acting horizontally, if $W=12$ lbs., and the base be to the length as 4 is to 5, find P.

7. The power being horizontal, if $W=48$ lbs., and the base be to the height as 24 is to 7, find P and R.

8. The power acting horizontally, if $R=2$ lbs., and $P=1$ lb., find W and the inclination of the plane.

9. The length of an inclined plane is 5 ft., and the height 3 ft. Find into what two parts a weight of 104 lbs. must be divided, so that one part hanging over the top of the plane may balance the other resting on the plane.

10. When a certain inclined plane ABC, whose length is AB, is placed on AC as base, a power of 3 lbs. can support a weight of 5 lbs.; find the weight which the same power could support if the plane were placed on BC as base, AC being then the height of the plane, the power in both cases acting along the plane.

11. If the power which will support a weight, when acting along the plane, be half that which will do so acting horizontally, find the inclination of the plane.

12. In the inclined plane, the power acting parallel to the plane, determine the relation between P and W.

Deduce the same also from the triangle of forces.

A string, inclined upwards at an angle of 30° to a plane whose inclination to the horizon is 30°, is just strong enough to support a weight W. Shew that if the string be parallel to the plane, it will be strong enough to sustain a weight $\frac{2W}{\sqrt{3}}$.

13. When a given weight is sustained on a given inclined plane by a force in a given direction, find the pressure on the plane.

Given the weight, and the magnitude and direction of the sustaining force, find the angle of the plane.

14. Find the angle at which a given force must act in order that it may just support a given weight on a given inclined plane.

15. If R be the pressure on the plane when the power acts horizontally, and R' when it acts parallel to the plane, shew that $W^2 = RR'$.

16. The inclination of a plane is 30°; a particle is placed at the middle point of the plane, and is kept at rest by a string passing through a groove in the plane, and attached

to the opposite extremity of the base; shew that the tension of the string is equal to the weight of the particle.

17. A power P acting along a plane can support W, and, acting horizontally, can support W'; shew that

$$P^2 = W^2 - W'^2.$$

18. On an inclined plane the pressure, force and weight are as the numbers 4, 5, 7; find the inclination of the plane to the horizon, and the inclination of the force's direction to the plane.

19. A weight W is sustained on an inclined plane by three forces, each equal to $\frac{P}{3}$, one acting vertically upwards, another parallel to the horizon, and a third lying between these and parallel to the plane; find the plane's inclination.

20. A weight W would be supported by a power P acting horizontally, or by a power Q acting parallel to the plane; shew that

$$\frac{1}{Q^2} = \frac{1}{P^2} + \frac{1}{W^2}.$$

21. A weight W is sustained on an inclined plane by a force P acting by means of a wheel and axle, placed at the top, in such a manner that the string attached to the weight is parallel to the plane. Given R and r, the radii of the wheel and axle, find the inclination of the plane.

22. A weight on an inclined plane may be raised either by a force along the plane, or by means of a system of pulleys of the first kind, with two movable pulleys, which will cause the power to act at an angle of 60° to the plane. Determine the means to be employed with most advantage.

23. A force which will just support a weight of 120 lbs. by means of three pulleys of the third system, is applied in the most advantageous manner to support a weight W on

a plane inclined to the horizon at an angle whose tangent is $\frac{3}{4}$; find the magnitude of W.

24. Two weights sustain each other on two inclined planes having a common altitude, by means of a string, parts of which are parallel to each of the planes; compare the pressures on the planes.

25. A weight w is supported on an inclined plane by two forces, each equal to $\frac{w}{n}$, one of which acts parallel to the base. Shew that equilibrium may be possible when the inclination of the plane is not greater than $2 \tan^{-1}\left(\frac{1}{n}\right)$, n being a positive integer.

26. A triangle ABC is placed with its plane vertical and its side BC in contact with the line of greatest slope of an inclined plane; shew that if prevented from sliding, it will topple over, if the angle of the inclined plane be greater than

$$\tan^{-1}\frac{a + c\cos B}{c\sin B}.$$

27. An inclined plane being free to move about its lower edge, and a weight being supported on it by a string which, being parallel to the plane, passes through a small smooth ring attached to the top of the plane, and then passing over a smooth fixed pulley supports a weight at its other end; shew that when there is equilibrium, $p = d\cot\alpha$, where d is the distance of the first weight from the foot of the plane, p the perpendicular from the foot on the inclined portion of the string, and α the inclination of the plane to the horizon.

28. Two weights sustain each other upon two inclined planes having a common altitude by means of a string which is attached to each; find their position, taking into account the weight of the string, which is supposed to be uniform.

29. A uniform beam of given length and weight, rests with one end on a given inclined plane, and the other attached to a string AFP passing over a pulley at F given in position. Having given the weight P attached to the other end of the string, find the position in which the beam rests.

30. A smooth cylinder is supported on an inclined plane with its axis horizontal, by means of a string which, passing over the upper surface of the cylinder, has one end attached to a fixed point and the other to a weight W which hangs freely; if α be the inclination of the plane to the horizon, and θ the inclination to the vertical of that part of the string which is fastened to the fixed point, the weight of the cylinder is

$$\frac{2\ W \cos \frac{1}{2}\theta \sin (\frac{1}{2}\theta - \alpha)}{\sin \alpha},$$

θ being measured from the upper part of the vertical through the point to which the string is attached.

The Screw.

1. What must be the length of a lever at the extremity of which a pressure of 1 lb. will support a weight of 1000 lbs. on a screw, the distance between two contiguous threads threads being $\frac{3}{4}$ inch?

2. The tangent of the angle of a screw is $\frac{1}{4}$, the radius of the cylinder 4 in., and the length of the power arm 2 ft.; find the ratio of W to P.

3. Supposing that $5\frac{1}{2}$ turns cause the head of a screw to advance $\frac{2}{3}$ in., what power applied at the extremity of an arm 18 in. long will be required to produce a pressure of 1000 lbs. upon the head of the screw?

4. What force must be exerted to sustain a ton weight on a screw, the thread of which makes 158 turns in the

course of 12 in., and which is acted on by an arm 6 ft. long?

5. Find the inclination to the horizon of the thread of a screw which, with a pressure of 5 lbs. acting at an arm 2 ft. long, can support a weight of 300 lbs. on a cylinder of 3 in. radius.

6. The circumference of the circle corresponding to the point of application of P is 6 ft.; find how many turns the screw must make on a cylinder 2 ft. long, in order that W may be equal to 144 P.

7. The interval between the threads of a screw being $\frac{1}{4}$ in., and the diameter of the cylinder $\frac{3}{8}$ in., find the length of the thread in 14 revolutions.

8. If a screw make m turns in a cylinder a foot long, and the length of the power arm be n ft., find the mechanical advantage.

9. The length of the power arm is n times the radius of the cylinder; find the pitch of the screw that the mechanical advantage may be n.

10. The angle of a screw is 30°, and the length of the power arm is n times the radius of the cylinder; find the mechanical advantage.

11. Shew that the mechanical advantage in the case of the screw is the same as that obtained by a horizontal power acting through a lever on a weight supported on an inclined plane, the arm of the power of the lever being to the arm of the weight as the power arm of the screw is to the radius of the cylinder, and the inclination of the plane being the same as the pitch of the screw.

12. Prove that the power is to the whole pressure on the thread in a ratio compounded of the ratio of the interval between the threads to the length of the thread in one turn of the screw, and of the ratio of the radius of the screw to the radius of the circle described by the power.

Virtual Velocities.

1. In the first system of pulleys, if there be five strings at the lower block, and W rise through 6 in., find how much P will descend.

2. In the second system of pulleys, the distance of the highest pulley from the fixed end of the string which passes round it is 16 ft.; find the greatest height through which the weight can be raised, there being four pulleys.

3. In the first system of pulleys, if P descend through 1 ft., W will rise through $\frac{6}{n}$ inches, where n is the number of pulleys on the lower block.

4. In the first system of pulleys, if P descend through 12 feet while W rise through 2 feet, find the number of strings at the lower block.

5. Assuming the statical principles demonstrated in previous chapters, shew that the principle of virtual velocities holds on the inclined plane when the power acts horizontally, and when it acts at a given angle to the plane.

6. Employ the principle of virtual velocities to determine the relation between P and W on the inclined plane when the power acts horizontally, and when it acts at a given angle to the plane.

7. If two forces P, Q, acting at a point, have a resultant R, and the point receive a displacement so that p, q, r are the virtual velocities of these forces, prove

$$P\,p + Q\,q = R\,r.$$

8. Investigate, by means of the principle of *virtual velocities*, the conditions of equilibrium on the common wheel and axle, and on the differential axle. In converting the former into the latter by increasing the diameter of part of the axle, how far may the increase be continued without loss of advantage? Examine the effect of coiling the rope over both axles in the same direction.

9. A weight is supported on an inclined plane by a power acting through a wheel and axle placed at the top of the plane, the string attached to the weight being parallel to the plane and passing over the axle. Apply the principle of virtual velocities to determine the relation between P and W.

10. Apply the principle of virtual velocities to the following: In the screw, if the arm of the power be six times the radius of the cylinder, and the thread be wound at an angle of 30° to the axis, what power will sustain a weight of three tons?

11. By the principle of virtual velocities find the relation between P and W on a single movable pulley, the parts of the string not being parallel.

12. Apply the principle of virtual velocities to determine the ratio of the power to the weight, when the weight slides along a smooth vertical rod, and is attached by an inextensible string to a point in the rod, while the power acts horizontally at the middle point of the string.

Work.

1. What is the distinction between the "mechanical efficiency of an agent," and the "mechanical advantage of a machine?"

A weight is raised through 100 ft. ten times in an hour by a force of 1 lb. acting through a machine whose advantage is 33; find the equivalent "horse-power."

2. What should be the horse-power of a steam engine capable of dragging 600 tons of stone per day of 10 hours up a smooth inclined plane of length 100 yards, and inclination 30°?

What if friction acts, coefficient of friction being .6?

3. A well forty feet deep, six feet diameter, and half full of water, is to be emptied by an endless chain of buckets,

each of which is one foot deep and two feet diameter; find the *work* to be done, and shew that a ten-horse-power engine would do it in little more than three minutes, disregarding the weight of the buckets, friction and resistances, and noting that a cubic foot of water weighs 1000 ounces.

Friction.

1. Find the coefficient of friction if a weight just rest on a rough plane inclined to the horizon at an angle of 30°.

2. If the height of a rough inclined plane be to the length as 3 is to 5, and a weight of 15 lbs. be supported by friction alone, find the force of friction in lbs.

3. A body placed on a horizontal plane requires a horizontal force equal to its own weight to overcome the friction; supposing the plane gradually tilted, find at what angle the body will begin to slide.

4. A weight of 10 lbs. just rests on a rough plane inclined at an angle of 45° to the horizon; find the pressure at right angles to the plane, and the force of friction exerted.

5. A force of 3 lbs. can, when acting along a rough inclined plane, just support a weight of 10 lbs., while a force of 6 lbs. would be necessary if the plane were smooth; find the resultant pressure of the rough plane on the weight.

6. A body is supported on a rough inclined plane by a force acting at a given angle to the incline. Shew that the reaction will be greatest when the body is just on the point of moving down the plane, and least when just on the point of moving up.

7. A body is just kept by friction from sliding down a rough plane inclined at an angle of 30° to the horizon; shew that no force acting along the plane would pull the body upwards unless it exceeded the weight of the body.

8. A weight of 10 oz. is just supported on a rough plane whose inclination is 75° by a power of 5 oz. acting parallel to the plane; find the inclination of the plane on which the weight would just rest of itself.

9. A uniform ladder 10 ft. long rests with one end against a smooth vertical wall and the other on the ground, the coefficient of friction being $\frac{1}{2}$; find how high a man whose weight is four times that of the ladder may rise before it begins to slip, the foot of the ladder being 6 ft. from the wall.

10. A beam rests with one end on the ground, and the other in contact with a vertical wall. Having given the coefficients of friction for the wall and the ground, and the distances of the centre of gravity of the beam from the ends, determine the limiting inclination of the beam to the horizon.

11. A cylinder whose altitude is double the diameter of the base is placed on an inclined plane; find what must be the coefficient of friction so that it may be just on the point of sliding and overturning.

12. A right circular cylinder of radius one-sixth of its length, standing on a rough inclined plane, will topple over just as it begins to slide; find the coefficient of friction.

13. A given force P, acting parallel to the horizon, just sustains a body of given weight W, on a rough inclined plane, the angle of which is θ. The same body will just rest without support on a plane of the same material, the inclination of which is α. Determine θ.

14. A uniform rod is held at a given inclination to a rough horizontal table by a string attached to one of its extremities, the other extremity resting upon the table; find the least angle at which the string can be inclined to the horizon without causing the rod to slide along the table.

15. Find the work done by a traction of 20 lbs., acting at an angle of 30° to the horizon on a body which moves along a rough level plane through a distance of 100 yards.

Determine the coefficient and the amount of friction in this case, the weight of the body being 50 lbs.

16. A body is kept at rest on a given inclined plane by a force making a given angle with the plane; shew that the reaction of the plane when it is smooth is a harmonic mean between the greatest and least reactions when it is rough.

17. Two inclined planes are joined at their vertex, their angles of inclination being 30° and 60° respectively; find the relation between their coefficients of friction, that two equal weights of the same substance may be at rest when supported on them by a string passing over the vertex.

18. Two inclined planes of the same altitude, one of which is twice as long as the other, the angles of inclination being complementary, are joined at the top; find what must be the relation between the coefficients of friction for the planes, so that two equal heavy particles, placed one on each plane, and connected by a cord passing over a smooth pulley at the top, may be just in equilibrium.

19. $O\,A$ is the vertical radius of a rough quadrant $A\,B$; P and Q are two weights connected by a string, and P hangs freely along $A\,O$, while Q rests on the convex side of the quadrant. Prove that the difference between the greatest and least values of P consistent with equilibrium is $2\,\mu\,Q \cos\alpha$, α being the angle $A\,O\,Q$, and μ the coefficient of friction.

20. A body resting on a rough plane of inclination α, is just on the point of moving up the plane under the influence of a horizontal force equal to its own weight; the plane is then tilted until the body is on the point of sliding down, the power remaining horizontal and having the same magnitude; find the inclination of the plane when this takes place.

21. Two uniform rods, each of length $2a$, are fastened together so as to form two sides of a square. One of them rests on the top of a rough peg. Shew that the distance of the point of contact from the middle point of the rod is $\frac{a}{2}(1-\mu)$, μ being the coefficient of friction.

22. An isosceles wooden trestle, consisting of two legs hinged at the top, standing on a rough floor, will support a weight only when the legs include an angle less than 30°. It is made to support a given weight on the same floor when the legs include an angle of 60° by connecting the feet by a string; determine the limiting values of the tension of the string and the thrust of a leg, neglecting the weight of the trestle.

23. A uniform rod of weight W rests with one end against a rough vertical plane, and with the other end attached to a string which passes over a smooth pulley vertically above the former end, and supports a weight P. Find the limiting position of equilibrium, and shew that equilibrium will be impossible unless P be not less than $W \cos \epsilon$, $\tan \epsilon$ being the coefficient of friction.

24. A uniform heavy rod having one extremity attached to a fixed point, passes over the circumference of a rough ring whose centre is at the fixed point and plane inclined at a given angle to the vertical; find the limiting position of equilibrium.

25. To a weight Q, placed on a rough horizontal plane (coefficient of friction $\cot \alpha$), is attached a string which passes over a pulley C, then over Q, and hanging vertically, supports a weight P. If θ be the angle which CQ makes with the horizontal plane, shew that in the limiting position of equilibrium

$$\sin(\alpha + \theta) = \frac{P+Q}{2P} \cos \alpha,$$

neglecting the friction between the string and Q.

26. A sphere of radius a is supported on a rough inclined plane (for which the coefficient of friction is μ) by a string of length $\frac{a}{\mu}$, attached to it and to a point in the plane. Prove that the greatest possible elevation of the plane, in order that the sphere may rest when the string is a tangent, is $2 \tan^{-1} \mu$; and find the tension of the string and the pressure on the plane in the limiting position of equilibrium.

27. If the ratio of the greatest to the least force which, acting parallel to a rough inclined plane, can support a weight on it, be equal to that of the weight to the pressure on the plane, the coefficient of friction will be $\tan \alpha \tan^2 \frac{1}{2} \alpha$, where α is the inclination of the plane to the horizon.

28. A heavy body is to be conveyed to the top of a rough inclined plane, the angle of inclination being α; prove that if the coefficient of friction be greater than

$$\frac{\sin\left(\frac{\pi}{4} - \frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)},$$

it will be easier to lift the body than to drag it up by means of a cord parallel to the plane.

29. A uniform and straight plank of length $2a$ and weight W, rests with its middle point on a rough horizontal cylinder (radius c) which is fixed, their directions being perpendicular to each other. Find the greatest weight that can be put upon one end of the plank without its sliding off the cylinder.

30. A weight supported on an inclined plane by a force acting at an angle θ to the plane, is on the point of moving down the plane when its inclination is α, and up the plane when its inclination is β; if the coefficient of friction be $\tan \psi$, prove that

$$\tan \theta \tan \psi = \cot\left(\frac{\alpha+\beta}{2}\right) \tan\left(\frac{\beta-\alpha}{2} + \psi\right).$$

DYNAMICS.

CHAPTER I.

1. A boat is sailing east at the rate of 15 miles per hour, and another boat is observed in the same track to be sailing relatively to the former at the rate of 10 miles per hour west. What is the actual motion of the latter?

2. A railway train travels over 150 miles in 5h. 40m. What is its average velocity, a foot and a second being the units of space and time?

3. A point passes uniformly over 100 yards in 30 minutes; what is the numerical representation of its velocity, according to the usual conventions respecting the units of space and time?

4. A man walks with a velocity represented by 3, and he finds that he walks 5 miles in 2 hours; if 1 foot be the unit of space, what is the unit of time?

5. A body whose velocity is represented by 3, moving uniformly, describes 64 ft. in 10 seconds; the unit of space is 1 foot; what is the unit of time? By what would its velocity be represented, if the units of space and time were 3 yards and 2 minutes?

6. A point is observed to describe uniformly a feet in n seconds; supposing the unit of time to be 1 minute, what must be the unit of space in order that the numerical value of the point's velocity may be 1?

7. If a shilling be the unit of money, £1000 a year the unit of income, and an inch per minute the unit of velocity, find the unit of space.

8. If 1 yard be the unit of space, and 1 minute the unit of time, what will be the numerical value of an acceleration of 32 feet per second?

9. If f be the measure of a uniform acceleration when a foot and a second are the units of space and time, find its measure when a mile and an hour are the units.

10. If f, F be the numerical values of any the same acceleration referred to units of time and space t, s; τ, σ, respectively, shew that $F = \frac{s}{\sigma}\left(\frac{\tau}{t}\right)^2 f$.

11. If f_1, f_2 be the measures of an acceleration when $m+n$ and $m-n$ seconds are the respective units of time, and a and b the respective units of distance, shew that the measure becomes $\frac{1}{c}\left(\sqrt{f_1 a} + \sqrt{f_2 b}\right)^2$, provided 2 m seconds be the unit of time, and c the unit of distance.

12. Find the numerical value of the acceleration when in half a second a velocity is produced which would carry a body over four feet in every quarter of a second.

13. A body moving with uniform acceleration describes 40 ft. in the half second which follows the first second of its motion; find the acceleration.

14. Discuss the whole motion of a particle which has a velocity of 100 at a certain point, and suffers a retardation of 10 during each unit of time.

15. A particle moving with a velocity 5 ft. per second is uniformly retarded, and after 30 seconds is found to be moving with the same velocity in the opposite direction; find the amount of the retardation, and express it in yards per minute.

16. A body moving from rest is observed to move over a feet and b feet respectively in two consecutive seconds; shew that the acceleration is $b-a$, and find the time from rest.

17. Uniform acceleration is defined as that which generates equal velocities in equal *times;* would it be correct to define it as that which generates equal velocities while the body moves through equal spaces?

18. Shew that in the formula $s = ut - \frac{1}{2} ft^2$, s will not represent the space described when the time elapsed is greater than $\frac{u}{f}$.

19. In the formula $s = ut - \frac{1}{2} ft^2$, discuss the motion as t increases, and explain the two values of t for a given value of s.

20. If u be the initial velocity, f the acceleration, v the velocity at time t, and s the space described, shew that

$$\frac{v-u}{f} = \frac{2s}{v+u} = t.$$

21. A particle projected in the direction of a uniform acceleration, describes P and Q feet in the pth and qth seconds respectively. Find the magnitude of the acceleration, and the velocity of projection.

22. If u be the initial velocity of a particle, and v its velocity after a time t, the motion being uniformly accelerated, find by general reasoning through what space the particle must move from rest to acquire the velocity $v - u$.

23. A particle moving with a certain velocity is observed n seconds afterwards to be moving with the same velocity in the opposite direction; shew that the space passed over in the interval is half that which would have been described if the particle had moved uniformly.

24. A body describes 72 ft. while its velocity increases from 16 to 20 ft. per second; find the whole time of motion and the acceleration.

25. The velocity of a uniformly accelerated particle on passing a given point P is 10 feet per second; at a distance of 15 feet from P it is 20 feet per second; find its acceleration—also its distance from P when at rest.

26. Two bodies uniformly accelerated in passing over the same space have their velocities increased from a to b and from u to v respectively; compare their accelerations.

27. Having given the velocities at two points in the path of a particle whose motion is uniformly accelerated, f being the acceleration, determine the distance between them.

28. A particle is moving at a certain instant at the rate of 10 ft. per second: after 4 seconds it is moving at the rate of 600 yards per minute, and after 16 seconds more at the rate of one mile in 48 seconds. Shew that, so far as your information goes, the particle may be moving with uniformly accelerated motion, and assuming that this is the case, find the numerical value of the acceleration.

29. A stone is dropped into a well, and after 2 seconds is heard to strike the water; required the depth to the surface of the water, the velocity of sound being 1132 ft. per second, and the uniform acceleration of the stone's motion 30 ft. per second.

30. A body starts from rest under a uniform acceleration, but at the commencement of each successive second the acceleration is decreased in a geometrical proportion ($r = \frac{1}{2}$); shew that the space described in n seconds $= \left(2n - 3 + \frac{3}{2^n}\right) 2s$, where s is the space described in the first second.

Parallelogram of Velocities.

1. A particle is moving with such a velocity, and in such a direction, that the resolved parts of its velocity in the direction of two lines perpendicular to each other are respectively 3 and 4; determine the direction and velocity of the particle's motion.

2. If v, v' be two component velocities of a particle, and α the angle between their directions, the resultant velocity is $\sqrt{(v^2 + v'^2 + 2vv' \cos \alpha)}$.

3. A particle is proceeding with uniform velocity along a certain straight line, and another velocity is communicated to it such that it moves along a line making 60° with its former direction and with unaltered velocity. Determine the magnitude and direction of the communicated velocity.

4. If velocities be separately communicated to a particle on the circumference of a circle, such that they would respectively cause the particle to move to the circumference along chords at right angles to each other in the same time, shew that when the velocities are communicated to it simultaneously, the time of its reaching the circumference will be unaltered.

5. If a particle has a velocity of 20 where a day and a mile are the units of time and space, and a velocity perpendicular to the former of $2\frac{4}{15}$ when the units are a minute and a foot, find the resultant velocity when an hour and a yard are the units.

6. A boat is rowed across a river (2 miles wide) in a direction making an angle of 60° with the bank and against the stream. The boat travels at the rate of 5 miles per hour, and the river flows at the rate of three miles per hour. Find at what point of the opposite bank the boat will land.

7. In the preceding problem, at what angle to the bank should the boat be rowed in order to land at the opposite point of the river?

8. If a be the distance between two moving points at any time, V the resultant of their velocities combined, u, v the resolved parts of V in and perpendicular to the direction of a, shew that their distance when nearest each other is $\frac{av}{V}$, and the time of arriving at this distance is $\frac{au}{V^2}$.

9. Two vessels are sailing with velocities V and V' in directions which make an angle α with each other; if O be

the angle between the line joining them and the direction in which a gun must be fired from one of them in order to strike the other, find θ for any given position of the vessels, and shew that its greatest value is given by the equation $v \sin \theta = (V^2 + V'^2 - 2VV' \cos \alpha)^{\frac{1}{2}}$, where v is the velocity of the shot, supposed uniform.

10. If a point be situated at the intersection of the perpendiculars drawn from the angular points of a triangle to the sides respectively opposite to them, and have component velocities represented in magnitude and direction by its distances from the angular points of the triangle, prove that its resultant velocity will tend to the centre of the circle about the triangle, and will be represented by twice the distance of the point from the centre.

CHAPTER II.

Mass.

1. If a body weighing 30 lbs. be moved by a constant force which generates in it in a second a velocity of 50 feet per second, find what weight the force would statically support.

2. A body weighing n lbs. is moved by a constant force which generates in the body in one second a velocity of a feet per second; find the weight which the force could support.

3. Shew that a force 100 times the difference in the weight of a unit mass at Toronto and New York will generate in the unit mass a velocity of one foot per second, the value of g being 32·17 in Toronto, and 32·16 in New York.

4. If the unit of pressure (or statical force) be 1 lb., and the unit of accelerating force be the force which in a second generates a velocity of one foot per second, what is the unit of mass?

5. In the relation $P = m f$, supposing the unit of force to be 5 lbs., and the unit of acceleration, referred to a foot and a second as units, to be 3, find the unit of mass.

6. If $P = m f$ be the relation between the pressure, mass and acceleration, find the unit of time, when that of space is two feet, and that of pressure the weight of a unit of mass.

7. In the equation $W = m g$, what must be the relation between the units of time and space in order that the unit of mass may be the mass of a unit of weight?

8. A pressure P produces an accelerating effect f on a mass m; determine the relation between P, m and f, the unit of pressure being 1 lb., the unit of mass the mass of a cubic foot of water, and the unit of acceleration the acceleration produced by gravity.

9. In what time will a force which can support a weight of 5 lbs. move a mass of 10 lbs. weight through a space of 50 ft. on a smooth horizontal plane?

10. Find in what time a force which would support a weight of 4 lbs., would move a weight of 9 lbs. through 49 ft. along a smooth horizontal plane; and find the velocity acquired.

11. A mass of 5 lbs. is observed to move from rest through a distance of 10 feet in one second, and a mass of 6 lbs. through a distance of 25 feet in two seconds; compare the forces acting to produce these motions.

12. Find how far a force which would support a weight of n lbs., would move a weight of m lbs. in t seconds; and find the velocity acquired.

13. Two bodies urged from rest by the same uniform force describe the same space, the one in half the time the other does; compare their final velocities, and their momenta.

14. A body weighing 10 lbs. is falling under the action of gravity, and is being pressed vertically downwards with a pressure of 1 lb. by another body, such as the hand; determine its velocity at the end of 3 seconds from rest.

15. If a weight of 8 lbs. be placed on a horizontal plane which is made to descend vertically with an acceleration of 12 ft. per second, find the pressure on the plane.

16. If a weight of n lbs. be placed on a horizontal plane which is made to ascend vertically with an acceleration f, find the pressure on the plane.

17. If a force of a lbs. produce in a cubic foot of matter in one second a velocity of n ft. per second, find the specific gravity of the matter.

1. State Newton's three laws of motion, and apply them to a general explanation of the following cases:

(1) A heavy particle moving along a smooth horizontal plane, gravity always acting perpendicular to the plane.

(2) A heavy particle moving down smooth inclined planes of different inclinations.

(3) Different heavy particles moving down the same inclined plane.

2. Particles are projected simultaneously from a point with equal velocities, and at the same elevation, and are subject to the action of gravity; prove that at any subsequent moment they will lie on a horizontal circle.

3. Shew from the second law of motion that if a system of particles subject to gravity be projected simultaneously from a point in directions which all lie in one plane, the locus of the particles at any subsequent instant will be a parallel plane.

Free Rectilineal Motion.

1. What is the character of a force which produces a uniformly accelerated motion?

State the modification to be introduced into the formula $s = ut + \frac{1}{2} ft^2$, so as to include the case of a force acting in a direction opposite to the motion of the particle. Draw a diagram to illustrate this case geometrically in Newton's manner.

2. A particle is moving under the action of a uniform force, the accelerating effect of which is f; if u be the arithmetic mean of the first and last velocities in passing over any portion h of the path, and v the velocity gained, shew that $uv = fh$.

3. If a quarter of a second be the unit of time, what must be that of space, that the acceleration of a particle falling under the action of gravity may be the unit of acceleration?

4. If the expressions for the accelerating effects of two forces, referred to t_1'' and t_2'' respectively, as units of time be equal; then if f and f' be the expressions for the accelerating effects of the two forces, referred to the same unit of time, $f : f' :: t_2 : t_1$.

5. A particle falls through the same distance at two different places on the earth's surface, and it is observed that the time of falling is t'' less, and the velocity acquired m ft. greater at one place than at the other. If g, g' be the accelerations of gravity at the two places, shew that $g\,g' = \frac{m^2}{t^2}$.

6. A constant force (f) acts upon a body from rest during 3 seconds, and then ceases. In the next 3 seconds it is found that the body describes 180 ft. Find both the velocity (v) of the body at the end of the 2nd second of its motion, and the numerical values of the accelerating force (1) when a second, (2) when a minute, is taken as the unit of time.

7. A body is projected vertically upwards with a velocity which will carry it to a height $2g$; find after what interval the body will be descending with the velocity g.

8. A point moving with uniform acceleration describes 20 ft. in the half second which elapses after the first second of its motion; compare its acceleration f with that of a falling heavy particle, and give its numerical measure, taking a minute as the unit of time, and a mile as that of space.

9. A body falling *in vacuo* under the action of gravity is observed to fall through 144·9 ft. and 177·1 ft. in two successive seconds; determine the accelerating force of gravity, and the time from the beginning of the motion.

10. A stone is thrown vertically upwards, and is observed to be at a height of twenty feet after three seconds. How much higher will it ascend?

11. A particle is projected vertically upwards, from a height of 257·6 ft., with a velocity of 96·6 ft.; find the time of its reaching the ground, and the momentum with which it impinges.

12. A particle projected from the deck of a vessel which is moving uniformly at the rate of 14 miles per hour, rises to a height above the deck of 64·4 feet; find how far the vessel will have gone before the particle again comes to the level of the deck.

13. Two balls are projected at the same instant towards each other, from the extremities of a vertical line, each with the velocity that would be acquired in falling down it. Where will they meet?

14. At the same instant one body is dropped from a given height, and another is started vertically upwards from the ground with just sufficient velocity to attain that height; compare the time they take to meet with the time in which the first would have fallen to the ground.

15. A particle is let fall from a tower 200 ft. high. Shew that the velocity with which another must be projected vertically downwards from the same point, two seconds afterwards, so as to reach the ground at the same instant, is 105·3 ft. per second, nearly.

16. A ball weighing 30 lbs. is projected along a horizontal surface with an initial velocity of 50 ft. per second. Its motion is uniformly retarded, and it is reduced to rest at a distance of 250 feet from the point of projection. What is the value of the retarding moving force?

17. A balloon ascends with a uniformly accelerated velocity, so that a weight of 1 lb. produces on the hand of the aeronaut sustaining it a downward pressure equal to that which 17 oz. would produce at the earth's surface. Find the height which the balloon will have attained in one minute from the time of starting, not taking into account the variation of the accelerating effect of the earth's attraction.

18. A particle falls from rest under the action of gravity, and the whole time of falling is divided into any number of intervals, during each of which it falls through the same

distance; shew that the accelerations of the motion during the mth and nth of these intervals are as

$$\sqrt{m} - \sqrt{m-1} : \sqrt{n} - \sqrt{n-1}.$$

19. A particle moving under the action of a uniform force in a straight line is observed at a point B; two seconds previously it was seen to be at A, where $AB = 10$ feet, and was then moving with a velocity 6, but in which direction was not observed. Shew that its motion may have arisen from a force in direction AB which would produce an acceleration 11, or from one in direction BA giving an acceleration 1; and the particle either was at B $2\frac{10}{11}$ seconds previously, or will be there again after 8 seconds.

20. A particle moves from rest at A, under the action of a constant force f, towards B; when it arrives at B, the force instantly changes direction, and again when it reaches A, and so on. Shew that when it passes through B the nth time its (velocity)^2 will be $f.\,AB\left\{2n - 1 - (-1)^n\right\}$, and that the particle then comes to rest at a distance $\left\{2n + 1 - (-1)^n\right\}\frac{AB}{2}$ from A.

Motion on Inclined Planes.

1. Find the velocity acquired by a particle in sliding down a plane of length 4 ft., inclined at an angle of 30° to the horizon.

2. Shew how to place a plane of given length in order that a body may acquire a given velocity by falling down it.

3. A heavy particle slides down a smooth plane of given height; prove that the time of its descent varies as the secant of the inclination of the plane to the vertical.

4. If a particle be projected vertically upwards at a height of a feet, with the velocity which it would have acquired in falling down a smooth plane of length $2l$, inclined to the

vertical at an angle of 45°, find the height to which it will rise, the time when it will return to the point of projection, and when it will reach the earth.

If a were very large compared with the earth's radius, for what reason would your result not apply?

5. A particle being projected down an inclined plane with the velocity which would be acquired in falling down its perpendicular height, the time of descent is found to be that of falling down the height. Find the plane's inclination.

6. If a heavy particle be projected with a velocity of 10 feet per second up an inclined plane, which rises at the rate of 1 vertical in 3 horizontal, through what space will it have moved at the end of 5 seconds?

7. A heavy particle is projected directly up an inclined plane (inclination α) with velocity u, and is attached to the point of projection by an inextensible string, whose length is half the distance a free particle would ascend; determine the time which elapses before the particle returns to the point of projection.

8. If two heavy particles commence to descend along the sides of a triangle, whose plane is vertical and base horizontal, at the same instant that a third falls freely from the vertex, find geometrically the position of the former particles when the third has reached the base.

9. The time of descent of a particle down any chord of a vertical circle from the highest or to the lowest point is the same.

The velocity acquired in falling to the end of the chord in either case varies as the length of the chord.

10. The time of descent from a given point to the centre of a circle vertically below it is the same as that to the circumference down a tangent.

11. Find by a geometrical construction or otherwise the line of quickest descent,

(i.) From a given straight line to a given point.

(ii.) From a given point within a given circle to the circle.

(iii.) From a given circle to a given point within it.

(iv.) From a given circle to a given straight line.

12. APB, AQC are two circles with their centres in the same vertical line ABC, and touching each other at their highest points. If APQ, Apq be any two chords, the times of descent down PQ, pq, from rest at P and p, are equal.

13. Two bodies, A and B, descend from the same extremity of the vertical diameter of a circle, one down the diameter, the other down the chord of 30°. Find the ratio of A to B when their centre of gravity moves along the chord of 120°.

14. AB is the vertical diameter of a circle; through A, the highest point, any chord AC is drawn, and through C, a tangent meeting the tangent at B in the point T. Shew that the time for a body's sliding down $CT \propto \frac{1}{AC}$.

15. If SP, SQ be the lines of longest and quickest descent from a point S to points on the circumference of a circle standing in the same vertical plane with S, and SP, SQ meet the circle again in p, q, prove that the sum of the squares of the times down SP, SQ is equal to that for those down Sp, Sq.

16. A particle slides down a rough plane inclined at an angle α to the horizon; find the acceleration, μ being the coefficient of dynamical friction.

17. A body is projected up a rough inclined plane with the velocity $2g$; the inclination of the plane to the horizon is 30°, and the coefficient of friction is equal to tan 15°. Find the distance along the plane which the body will describe.

18. A heavy body is projected up a rough plane inclined at an angle of 60° to the horizon, with the velocity which it would have acquired in falling freely through a space of 12 feet, and just reaches the top of the plane; find the altitude of the plane, the coefficient of dynamical friction being $\frac{1}{\sqrt{3}}$.

19. AP, AQ are two inclined planes, of which AP is rough ($\mu = \tan PAQ$) and AQ smooth, AP lying above AQ; shew that if bodies descend from rest at P and Q, they will arrive at A, (1) in the same time if PQ be perpendicular to AQ, (2) with the same velocity if PQ be perpendicular to AP.

20. A body is projected up a rough inclined plane, the inclination of the plane to the horizon being α, and the coefficient of friction $\tan \epsilon$; if m be the time of ascending and n the time of descending, shew that

$$\left(\frac{m}{n}\right)^2 = \frac{\sin(\alpha - \epsilon)}{\sin(\alpha + \epsilon)}.$$

Projectiles.

1. A body is projected with a velocity $3g$ at an inclination of 45° to the horizon; determine the range.

2. What angle of projection gives the greatest range theoretically? Why does experiment contradict this result?

3. If the initial velocity of a projectile be given, the horizontal range is the same, whether the angle of projection be $\frac{\pi}{4} + \alpha$ or $\frac{\pi}{4} - \alpha$. Compare also the times of flight.

4. Point out what parts of Newton's second law of motion are illustrated in the motion of a common projectile.

Shew that the sum of the greatest heights in the two paths which have the same range is half the greatest range, the velocity of projection being given.

5. A body is projected with a vertical velocity (16·7), and a horizontal velocity (·8); prove that its distance from the point of projection at the end of one second is one foot.

6. A particle is projected with a horizontal velocity $12g$, and moves under the action of gravity. Find the position of the particle, and the direction and magnitude of its velocity after five seconds, pointing out clearly the dynamical principles of which you assume the truth.

7. The greatest elevation of a particle projected with a velocity of 80 feet per second is 50 feet; find the direction of projection.

8. Given two straight lines representing in magnitude and direction the horizontal and vertical velocities of a projectile at the point of projection, obtain a geometrical construction for the vertical velocity at a point where the direction of motion makes a given angle with the direction of projection.

9. During the motion of a heavy particle projected *in vacuo* with a velocity v and at an angle α ($> 30°$) to the horizon, the times when the whole velocity is double the vertical velocity are

$$\frac{v}{g}\left(\sin\alpha \pm \frac{1}{\sqrt{3}}\cos\alpha\right).$$

Interpret this result when $\alpha < 30°$.

10. Having given the velocities at two points of the path of a projectile, find the difference of their altitudes above a horizontal plane.

11. If two particles be projected from the same point in the same direction with different velocities, prove that the line joining them moves parallel to the direction of projection.

12. A body is projected vertically upwards from a point A with a given velocity (u); find the direction (α) in which another body must be projected at the same instant with a given velocity (v) from a point B in the same horizontal line with A, so as to strike the first body.

13. From the top of a tower two bodies are projected with the same given velocity (v), at different given angles of elevation (α, β), and they strike the horizon at the same place; find the height of the tower.

14. A ball is fired from the ground at an angle of 45°, so as just to pass over a wall 10 feet high at a distance of 260 feet. Shew that it will strike the ground 10·4 feet from the wall.

15. A particle is projected from the top of a tower 120 feet in height with the velocity which would be acquired by a particle falling freely $12\frac{1}{2}$ feet under the action of gravity, and in a direction inclined at an angle $\sin^{-1}\frac{3}{5}$ to the vertical. In its descent it just passes over a wall (perpendicular to the plane of projection) 30 feet in height. Shew that the particle will strike the ground 6 feet from the foot of the wall.

16. A plane is inclined at an angle of 45° to the horizon, and from the foot of it a body is projected upwards along the plane, and reaches the top with one-fifth of its original velocity (v); where will it strike the ground?

17. Swift of foot was Hiawatha;
He could shoot an arrow from him,
And run forward with such fleetness
That the arrow fell behind him!
Strong of arm was Hiawatha;
He could shoot ten arrows upward,
Shoot them with such strength and swiftness,
That the tenth had left the bowstring
Ere the first to earth had fallen.

Neglecting the resistance of the air, taking $g = 32$, supposing one second to elapse between the discharge of each of the ten arrows, and making Hiawatha shoot at his longest range, shew that he must have been able to run at least at the rate of 76 miles an hour.

18. In the motion of a projectile, prove that the direction of motion at any time t, when the horizontal and vertical velocities are u and v respectively, will intersect the plane at a distance $\frac{1}{2} g t^2 \frac{u}{v}$ from the point of projection.

19. A body is projected horizontally with a velocity $4g$ from a point whose height above the ground is $16g$; find the direction of motion (1) when it has fallen half way to the ground, (2) when half the whole time of falling has elapsed.

20. If a body be projected at an angle α to the horizon, with the velocity due to gravity in $1''$, its direction is inclined at an angle $\frac{\alpha}{2}$ to the horizon at the time $\tan \frac{\alpha}{2}$, and at an angle $\frac{\pi - \alpha}{2}$ at the time $\cot \frac{\alpha}{2}$.

21. If α be the angle of projection of a projectile, t the time which elapses before the body strikes the ground, prove that at the time $\frac{t}{4 \sin^2 \alpha}$, the angle which the direction of motion makes with the direction of projection is equal to $\frac{\pi}{2} - \alpha$.

22. In the path of a projectile, the velocities are v_1, v_2, at two points where the directions of motion are inclined to each other at an angle φ; T being the time of motion between these points, and v the velocity at the highest point, prove

$$v_1 v_2 \sin \varphi = g v T.$$

23. Two heavy bodies are projected from a point at the same instant with velocities v, v', and angles of elevation α, α'; shew that the distance between them will be parallel to a fixed direction, and will alter at the uniform rate

$$\left\{v^2 + v'^2 - 2 v v' \cos (\alpha - \alpha')\right\}^{\frac{1}{2}}.$$

24. A vertical circle stands on the ground, and particles are permitted to move down chords of this circle (considered

as smooth inclined planes) from the highest point to the circumference; prove that they all reach the circumference in the same time; and if each particle, after describing its chord, move freely to the ground, they will all strike the ground with the same velocity.

25. In the motion of a projectile, shew that the tangent of the inclination of the direction of motion to the plane at any time $= \frac{\text{vertical velocity}}{\text{horizontal velocity}}$.

If θ, φ be the angles made with the horizon by the tangents to the path of a projectile at the points P and Q, the time of describing the arc PQ is proportional to $(\tan\theta - \tan\varphi)$.

26. If $P, Q, R, \ldots$ be points on the path of a projectile, such that time in PQ = time in $QR = \ldots$, prove that the tangents of the angles made with the horizon by the directions of motion at the points $P, Q, R, \ldots$ are in arithmetic progression.

27. In the path of a common projectile, if the tangent be drawn to cut the horizontal and vertical lines passing through the highest point, one of the points of section moves with uniform velocity, and the other is uniformly accelerated.

28. Three heavy bodies, P, Q, R, are projected in a vertical plane at the same time from a point at angles of elevation α, β, γ, respectively, and so that the direction PR is always vertical, and QR horizontal; shew that the ratio of the rates at which the distances PR and QR change is

$$\frac{u \sec\gamma \sin(\alpha - \gamma)}{u\cos\alpha - v\cos\beta},$$

where u, v are the velocities of projection of P and Q respectively.

29. Find the range of a projectile fired horizontally on a plane inclined at an angle of $10°$ to the horizon.

30. The range of a projectile on a plane inclined to the horizon at an angle θ is the same for the two angles of elevation α and $90° - \alpha + \theta$.

31. Shew that the direction of projection which gives the greatest range upon *any* plane through the point of projection, bisects the angle between the vertical and the plane on which the range is measured.

32. A particle projected at an angle α to the horizon and with velocity v, impinges perpendicularly on an inclined plane drawn through the point of projection at an angle β to the horizon; shew that the range on the plane $= \frac{2v^2}{g} \cdot \frac{\cos\alpha \cos(\alpha - \beta)}{\sin 2\beta}$.

33. If a body fall down a plane of inclination α, and another be projected at the same instant from the starting point horizontally along the plane with velocity v, find the distance D between the two bodies (1) after a given time t, (2) after the first body has descended through a given space s.

34. A ball is fired at a given angle of elevation, and with a given charge, from the mast head of a ship sailing uniformly; prove that the distance of the ball from the mast on reaching the level of the deck is always the same; and shew that, the angle of elevation being α, this distance is

$$\frac{^2 \sin 2\alpha}{2g} \left\{ 1 + \sqrt{1 + \frac{2gh}{v^2 \sin 2\alpha}} \right\},$$

where v is the velocity of discharge, and h the height of the mast above the deck.

35. Two projectiles are shot from the same point, at the same instant, with the same velocity, and at angles which are complementary to each other; prove that the horizontal distance between them is always equal to the vertical distance; and that they will be moving in parallel directions after a time equal to the arithmetic mean between their times of flight.

36. A man standing on the edge of a ravine fires at a mark on the opposite bank in the same horizontal plane as his eye. His rifle is properly sighted for the distance, but he holds it in such a way that the plane through the line of

sight and the axis of the barrel is inclined to the vertical at an angle β. The ball strikes the opposite bank which is vertical in n seconds. Shew that it strikes at a distance $n^2g \sin^2 \frac{\beta}{2}$ below the mark, and $\frac{1}{2} n^2g \sin \beta$ to the right or left, the resistance of the air being neglected.

37. The barrel of a rifle sighted to hit the centre of the bull's-eye, which is at the same height as the muzzle, and distant a yards from it, would be inclined at an elevation α to the horizon. Prove that if the rifle be wrongly sighted, so that the elevation is $\alpha + \theta$, θ being small compared with α, the target will be hit at a height $\frac{a \cos 2\alpha}{\cos^2 \alpha}$. θ above the centre of the bull's-eye.

38. In the motion of a projectile shew that the focus of the parabola is above or below the horizon, according as the angle of projection is greater or less than that which produces the greatest range.

Shew that the angles of projection which cause the foci to be at equal distances above and below the horizon are complementary.

39. In the two trajectories of a common projectile, which for a given velocity of projection have the same range, the sum of the greatest heights is double the greatest height for the maximum range; and in this latter case the focus lies in the same horizontal plane as the point of projection.

40. In the motion of an ordinary projectile, shew that the distance of the focus of the parabola from the point of projection is independent of the angle of projection.

Falling Bodies Connected by a String.

1. Twelve pounds weight is so distributed at the extremities of a string passing over a pulley, that one end descends through 7 ft. in as many seconds. What weight is at each end of the string?

2. A weight of 10 lbs. is attached to one end of a string; find the weight which must be attached to the other, in order that when the system is suspended from a fixed pulley, the acceleration may be half that of gravity.

3. Two weights connected by a string passing over a pulley are in motion, and it is observed that the space through which the heavier descends in the nth second from the commencement of the motion is a feet; compare the weights.

4. In Atwood's machine, if the weight of each box be $31\frac{1}{2}$ oz., and an oz. weight be placed upon one, find the velocity generated in 5″.

If after a certain time the oz. weight be removed, and the space described during the next second be 4 ft., find the time from rest, and the whole space described ($g = 32$).

5. Two weights, P and Q, of 10 and 6 lbs. respectively, are suspended by a string over a smooth pulley. If after 12 seconds of motion P suddenly loses 7 lbs., find the time that will elapse before P returns to the point where it was thus diminished; and describe the subsequent motion, supposing P to take up again the weight it had lost.

What would have been the difference in the motion if instead of P losing 7 lbs., Q had taken up 16 lbs.; this latter weight being supposed previously at rest?

6. Two weights of 14 oz. and 13 oz. are attached by means of a string passing over a smooth pulley, and are allowed to move from rest for 3 seconds, when 9 oz. of the former become detached; shew that the remaining weights are momentarily reduced to rest in a quarter of a second.

7. P pulls Q over a smooth pulley; and Q in ascending, as it passes a certain point A, catches and carries with it a certain additional weight, and on its descent the additional weight is again deposited at A. Supposing no impulse to take place when the weight is so caught up, and that Q in this manner oscillates through an equal space on either side of A, find the additional weight.

8. Two unequal weights connected by a string which passes over a smooth fixed pulley, are in motion; shew that if at any instant the string breaks, when the ascending motion of one of them ceases, the other will have descended through three times the space through which the former has ascended in the interval.

9. A weight Q is drawn along a smooth horizontal table by a weight P hanging vertically; find (1) the acceleration of P, (2) the vertical and horizontal accelerations of the centre of gravity of P and Q.

10. Two bodies, P and Q, are connected by an inextensible string which passes over a smooth fixed pulley; find the tension of the string.

11. State what parts of the laws of motion are illustrated and confirmed by Atwood's machine.

Prove that the tension of the string remains always the same during the motion.

12. In Atwood's machine the tension of the string is the harmonic mean of the moving effects of gravity on the two bodies.

13. In Atwood's machine shew how the proportionality of the acceleration to the moving force on the same mass may be tested.

A weight of 10 lbs. hanging vertically, draws another of 7 lbs. down a smooth plane, inclined at an angle of 30° to the horizon. Find the vertical velocity of the latter at the end of 10 seconds from rest, and the tension of the string.

14. A uniform string hangs at rest over a smooth peg. Half the string on one side of the peg is cut off; shew that the pressure on the peg is instantaneously reduced to two-thirds its previous amount.

15. A body P descending vertically draws another body Q up the inclined plane formed by the upper surface of a

right-angled wedge which rests on a smooth horizontal table; find the force necessary to prevent the wedge from sliding along the table.

16. In Atwood's machine a weight $P\ (=\overline{n+1}\ p)$, starting from rest, draws up a weight $Q\ (=np)$, and at the end of successive intervals of time, each equal to t, P gains, and Q loses a weight p; shew that the velocities at the ends of these successive intervals are as the squares of the natural numbers, and that the space described in time nt

$$=\frac{n}{6}\cdot\frac{2n^2+1}{2n+1}\,gt^2.$$

17. Two bodies, whose weights are P and Q, hang from the extremities of a cord passing over a smooth peg; if at the end of each second from the beginning of motion P be suddenly diminished and Q suddenly increased by $\frac{1}{n}$th of their original difference, shew that their velocity will be zero at the end of $n+1$ seconds.

18. Two equal bodies connected by a string are placed upon two planes which are inclined at angles α, β, to the horizon, and have a common altitude. Prove that the acceleration of their centre of gravity is

$$g \sin \tfrac{1}{2}(\alpha-\beta) \cos^2 \tfrac{1}{2}(\alpha+\beta).$$

19. A fine inelastic thread is loaded with n equal particles at equal distances c from one another; the thread is stretched and placed on a smooth horizontal table, perpendicular to its edge, over which one particle just hangs; find the velocity of the system when the rth particle is leaving the table.

Hence shew that if a heavy string of length a be similarly placed on a horizontal table, its velocity in falling off will be $=\sqrt{ag}$.

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20. A heavy wedge rests upon a smooth horizontal plane, and a cannon is fixed in its face with its axis horizontal and perpendicular to the edge of the wedge; and in a direction

passing through its centre of gravity. A ball is fired from the cannon with an impulse equal to one-half of the vertical component of the blow with which it strikes the ground; and it just strikes the foot of the plane. Shew that if M, m be the masses of the wedge and ball respectively, the inclination of the face of the wedge to the horizon is $\tan^{-1}\frac{M}{M+m}$.

Impact.

1. A ball of 9 ounces, moving with a velocity of 7 feet a second, impinges directly upon a ball of 12 ounces, moving with a velocity of 5 feet a second in the opposite direction; find the change in the velocity and momentum of each ball, supposing them inelastic.

2. A perfectly elastic ball, moving from a point A in the circumference of a smooth circular billiard table of radius r, strikes the circumference n times in going round the table once before it reaches A; shew that its whole path is $2(n+1)\, r \sin\frac{180^\circ}{n+1}$.

3. What must be the elasticity of two balls A, B, in order that A, impinging directly upon B at rest, may itself be reduced to rest by the impact?

4. With what velocity must a ball impinge on an equal ball moving with a given velocity, that the impinging ball may be reduced to rest, the common elasticity of the balls being given?

5. Under what conditions will the velocities of two balls A, B, impinging directly upon one another, be interchanged after impact?

6. If the modulus of elasticity be $\frac{1}{3}$, at what angle must a ball be incident on a hard plane, that the angle between the directions before and after impact may be a right angle?

7. If u, u' and v, v' denote the velocities of two elastic bodies before and after impact, respectively, and w the common velocity at the instant when their velocities are equalized, prove

$$(u-w)(v'-w)=(w-u')(w-v).$$

8. Two inelastic balls of masses B and C moving with the same velocity v, but in opposite directions, after direct impact respectively with two balls each of mass A, moving with the same velocity and in the same direction, each move with a velocity u; shew that $\frac{B}{C}=\frac{u+v}{u-v}$, B being the ball overtaken by A, and the momentum of A before impact being greater than that of C.

9. When an elastic particle impinges on a fixed plane, what becomes of the lost momentum?

If particles are dropped from given heights upon a fixed horizontal plane, prove that if the heights are inversely as the squares of the elasticities, they all rise to the same height after reflection.

10. Two equal and perfectly elastic particles are started from given points in the same direction with given velocities, and one overtakes the other; after what time will the distance between them be the same as at first?

11. A particle whose elasticity is $\frac{1}{2}$, is dropped to the ground at the same instant that another is dropped from a point vertically above it at 9 times its height; shew that the latter impinges on the former just when it ceases to rebound.

12. A stone is dropped from a height, $36g$, at the same instant that another is thrown up from the point of the ground vertically beneath the former with a velocity of $6g$; prove that they will meet half way.

If the stones be equal and perfectly elastic, find the velocities which they will have on reaching the ground, and the time between their arrivals, the ball first arriving not being permitted to rebound.

13. An elastic ball, A, is projected vertically upwards from the foot of a line at the same time that another, B, is dropped from its top. They impinge directly; B's direction of motion is reversed, and it receives such a velocity as will just bring it to the top again. Shew that the velocity of A at projection was $2\sqrt{\frac{ag}{1+e}}$, where a is the length of the line and e the modulus of elasticity of the balls, their masses being equal.

14. An elastic ball is dropped from a height of a feet on a horizontal plane; shew that the distance passed over by it before it comes to rest will be $a\left(\frac{1+e^2}{1-e^2}\right)$, and the time will be $\sqrt{\frac{2a}{g}}\left(\frac{1+e}{1-e}\right)$, e being the modulus of elasticity of the ball.

15. If an elastic ball be projected at an angle α, and with velocity v, prove that the sum of all the horizontal ranges $= \frac{v^2 \sin 2\alpha}{g(1-e)}$.

16. If an elastic ball impinge successively against two adjacent sides of a rectangle, its velocity will be diminished in the ratio of $1 : e$, e being the modulus of elasticity.

17. Two elastic planes are inclined at an angle of 60°, and a ball is projected in a plane perpendicular to them both; and after reflection at each its direction is parallel to the first: shew that the angle of incidence at the first impact cannot be 30° whatever the modulus of elasticity may be; but that if the angle be less than 30°, a modulus can always be found such that the conditions of the problem may be satisfied.

18. A ball (elasticity e) is projected from a given point in the circumference of a circle; after being twice reflected at the circumference it returns to the point of projection. Required the direction of projection.

19. A ball of elasticity e is projected from a point on a billiard table, so that after reflection at the four sides of the table, it will return to the point of projection; shew that it will continue to describe the same path. Prove also that the successive velocities with which it leaves the same side are diminished in the ratio of $e^2 : 1$.

20. A ball impinges successively npon two fixed planes, inclined to each other at an angle of 45°. If the direction of motion before the first impact make with the normal at the point of contact, an angle $\tan^{-1}\frac{2}{3}$, and $e = \frac{1}{3}$, shew that after the second impact the direction of motion is perpendicular to the plane upon which the ball first impinged.

21. A ball impinges successively upon three fixed planes, placed in the form of an equilateral triangle. If the direction of motion before the first impact make with the normal at the point of contact an angle $\tan^{-1}\frac{1}{3}$, and $e = \frac{1}{\sqrt{3}}$, shew that after the third impact the direction of motion is inclined to the normal at an angle $\tan^{-1}\frac{\sqrt{3}}{2+\sqrt{3}}$.

22. If two equal and perfectly elastic particles be projected with velocities u, v in opposite directions from the same point within a thin smooth tube in the form of a circle, prove that if $\frac{u}{u+v} = \frac{m}{n}$ a fraction in its lowest terms, they will meet together at the point of projection at the nth impact.

23. If a perfectly elastic particle descend a chord of a vertical circle inclined at an angle θ to the vertical diameter, impinge on the tangent at the end of that diameter, and be there reflected, it will cut the circle at its point of greatest elevation if $\theta = \cos^{-1}\frac{1}{\sqrt{3}}$.

24. A perfectly elastic ball falls from the highest point of a vertical circle down equal chords in the form of tubes, rebounding from the tangents at their extremities; shew that the velocity acquired by the ball at the extremity of

any chord is equal to the velocity it would acquire there by moving down the chord from the highest point.

25. If a smooth billiard table have a smooth circular cushion, and an imperfectly elastic ball be impelled without rotation against the cushion with a velocity v in a direction making an angle θ with the normal to the cushion, find the values of v and θ after the nth contact, and shew that the ball will ultimately slide round the table in contact with the cushion with a velocity $v \sin \theta$.

26. A ball whose mass is m is projected from the ground with a velocity of 120 ft. per second in a direction inclined to the horizon at an angle whose tangent is $\frac{3}{4}$; and at the highest point of its path it comes in contact with another smooth ball whose mass is M, and which has been falling vertically through a height of 144 ft., the modulus of elasticity being equal to $\frac{m}{M}$. Shew that if the blow take place in the direction of M's motion, the distance between the points where the balls strike the ground will be independent of e, provided $m : M$ always equals e; and that if it take place in the direction of m's motion, the corresponding distance will be to that in the former case as e to 1.

27. A particle of elasticity e is projected in a plane at right angles to a smooth vertical wall from a point on the ground at a distance a from the wall; shew that after reflection at the wall, it will strike the ground again at a distance from the wall

$$= e\left(\frac{v^2 \sin 2\alpha}{g} - a\right),$$

v and α being the velocity and elevation of projection.

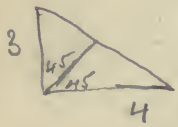
28. A shell is fired from a mortar on the ground, and after once ricochetting rises so that at its greatest altitude it just passes over a wall; if φ be the angle subtended by the wall at the mortar, shew that the angle of elevation (α) is given by

$$\tan \alpha = 2\frac{e+2}{e^2}\tan \varphi,$$

where e is the coefficient of elasticity between the ground and shell.

29. Two balls, A, B, are moving in directions at right angles to one another with the same velocity (u), the line joining their centres at the instant of impact being in direction of A's motion ; find the velocity and direction of motion of each after impact (elasticity $= e$).

30. A, B are two equal and perfectly elastic spheres ; A moving with a given velocity (u) impinges on B at rest, the direction of A's motion before impact making an angle of 60° with the straight line which joins their centres at the instant of impact ; determine the directions and velocities of A and B after impact.



ANSWERS.

STATICS.

CHAPTER I.—1. 8 lbs. 2. 3 ft. 4 in. 3. $\frac{Q}{P}a$ in. 4. $\frac{b}{a}P$ lbs. 5. As 4 to 3. 6. $\frac{b}{a}\cdot\frac{c}{d}p$, taking a unit of length to represent a unit of force.

CHAPTER II.—2. Complete parallelogram; then prop. is evident since diagonals bisect one another. 4. 5 lbs. 5. $\sqrt{3}\times$ less force. 6. $\sqrt{3}$, and acts perp. to 1. 7. Use first formulas in § 29; 4, 150°, 60°. 8. Evident on completing parallelogram, and drawing diagonal. 9. 120°; use second formulas, § 29. 10. Fces. in st. line. 11. $\mathrm{Cos}^{-1}-\frac{7}{12}$, *i.e.* the angle whose cosine is $-\frac{7}{12}$. 12. $5, 5\sqrt{3}$. 13. 3. 14. As $2:\sqrt{6}:\sqrt{3}+1$; § 29. 15. As $\sqrt{6}:2:\sqrt{3}-1$. 16. § 28, and Euc., Bk. I., Prop. 20. 17. In a line with the reversed force, and double it. 18. The sum or difference of the forces. 19. Components make angles 60°, 30°, with original force. 20. § 18, pressure on peg bisects angle between forces; resolving along this, pressure $=2\times$ force $\times\cos$ ($\frac{1}{2}$ angle between forces). 21. Press. on top peg $=w\sqrt{3}$; press. on side pegs $=\frac{w}{\sqrt{2}}(\sqrt{3}-1)$. 22. 0. 23. Apply two equal and opposite forces along direction of given force; this equivalent system can at once be resolved in the given directions. 24. Resolve the forces along and perp. to 1, say; result. is $\sqrt{3}$, and makes angle 30° with 3. 25. $2\sqrt{2}$, and bisects angle between 4 and 5. 26. $8\sqrt{4+2\sqrt{2}}$, and makes with 8 the angle $\tan^{-1}(1+\sqrt{2})$. 27. 180°. 28. The pt. is the bisection of the arc at the base of the segment. 29. We have only three indep. relations, viz., the first eq^{n}. of § 29 and $(Q,R)+(R,P)+(P,Q)=180°$. Angles being given, relative magnitudes of fces. are known. 30. Use prob. 2, and indirect proof. 31. O is the bisection of the line joining the bisections of AB, CD; and $PO:OQ::1:3$. 32. Indirect proof. 33. $\alpha>90°$, $P>R$. A figure makes second part evident. 34. Each diag. is equivalent to two sides, and, taking away the common side, opp. sides remain. 35. Diag. through C of parall^{m}. of which AC, BC are sides. 36. Let E be bisection of BC; then AB is result. of AC, CB; $2AE$ of AB, AC; and $3AD$ of AB, $2AE$. 38. Bisect AD in E; join EO (centre), and draw BEC perp. to EO. 39. On BC desc. seg. BOC cont^{s}. angle $=180-A$, and on AC seg. AOC cont^{s}. angle $=180-B$. O is pt., § 29. 40. By Geomet. $OD:OE:$

$OF = \cos A : \cos B : \cos C$; but, since O is at rest, $OD : OE : OF = \sin A : \sin B : \sin C$; $\therefore$ $\tan A = \tan B = \tan C$; and $A = B = C$. 42. According to order in which sides are taken. 43. Components make angles $22\frac{1}{2}°$, $67\frac{1}{2}°$ with side not repts. a force, and $= a\sqrt{1 + \frac{1}{\sqrt{2}}}$, a being a side.

Chapter III.—1. An area; a sq. in. 4. See notes. 5. Second pt., § 35. 7. 12 lbs. 8. Take mts. abt. pt. of attachment; tension of string disappears, and force $= \frac{W}{\sqrt{3}}$. 9. Take mts. abt. A; tension $= P\sqrt{3}$. 10. Mts. abt. fixed point. 11. 5 or 2 lbs. accd. as fulcrum is $\frac{1}{3}$ or $\frac{1}{8}$ length of lever from end. 12. 16, and acts at same pt. as 7. 13. Mts. abt. fixed pt. are equal. 14. $5 : \sqrt{6}$. 15. Mts. abt. fulcrum. 16. Incln. of AB to horizon $= \tan^{-1} \frac{P.AB + Q.BC \cos ABC}{Q.BC \sin ABC}$. 17. Incln. of BC to horizon $= \tan^{-1} \frac{(P+Q).BC + P.AB \cos ABC}{P.AB \sin ABC}$. 18. 45° = incln. of rope to horizon. 19. W. 20. If w = wt. of rod, W = suspended wt.; incln. of rod to horizon $= \tan^{-1} \frac{w + 2W}{2W}$. 21. $\frac{1}{3}$ length of rod from end. 22. $2a$, $2b$, lengths of longer and shorter arms; w_1, w_2 their wts.; required wt. $= \frac{aw_1 - bw_2}{2b}$. 23. Let ON be perp. from cr. on AB, W = wt. of rod, w = attached wt. Taking mts. abt. cr., reactions of circle disappear, and $W.ON \sin A = w.\frac{1}{2}AC$, $w = \frac{2W.ON}{\text{diam.}} \cdot \frac{BC}{AC} \propto \frac{BC}{AC}$. 24. Let R = reaction of wall, T = tension of string, W = wt. of rod, a = length of string, θ = incln. of rod to wall; taking mts. abt. pt. where rod touches wall, middle pt. of rod, and pt. of attachment of string to wall, $T a \cot \theta = W.ma \sin \theta$, $T(ma - a \cot \theta) = R.ma \cos \theta$, $W.ma \sin \theta = R.\frac{a}{\sin \theta}$; whence eliminate T, R, W. 25. $BD = BC$ or perp. to it. 26. If O be cr., dif. of mts. is expressed by 4 × area of OPQ. 28. $h\sqrt{30 + 12\sqrt{3}}$ × strength of one man. 29. Let $AB = a$, $BC = b$, angle bet. rod and string $= \theta$, wt. $= W$. Take mts. abt. A; press. $= 2W \cos \theta = W.\frac{\sqrt{4a^2 - b^2} + b\sqrt{8a^2 + b^2}}{a\sqrt{2}}$. 30. We obtain, taking mts. abt. cr. (1) $P \cos A + Q \cos B + R \cos C = 0$, (2) $P \sin A$ + &c. $= 0$; whence relns. If $B = C$, $R = -Q$, & $P = 0$; if $A = B = C$, $P + Q + R = 0$.

Chapter IV.—2. The posn. of cr. is same whatever be dirn. of forces; let them be perp. to pl. of triangle, and take mts. abt. BC. 4. Let fces. act at all the angles; cr. is cr. of fig.; now remove one

force by applying an equal and opposite force, and the posn. of the reqd. resultant is obtd. at dis. from cr. of fig. equal to $\frac{1}{6}$ length of side. 5. At dis. from cr. $= \frac{\text{rad.}}{n-1}$. 9. 1 ft. from end. 10. $2\frac{1}{2}$ ft. from end; changes $\frac{7}{24}$ ft. 11. 20 lbs. 12. 90 lbs. 13. Accd. as $\frac{x_1 w_1 + \ldots}{w_1 + \ldots} > \frac{x_1 w_1 + \ldots + w(x_1 + \ldots)}{w_1 + \ldots + nw} > \frac{x_1 w_1 + \ldots}{w_1 + \ldots}$; accd. as $\frac{x_1 w_1 + \ldots}{w_1 + \ldots} < \frac{x_1 w + \ldots}{w + \ldots}$. 15. Bisection of line joining bisections of opp. sides. 16. It must pass through intersn. of horiz. line through B, and vert. through cr. rod (Prob. 3, Ch. II.) 17. § 43, and that if two forces counterbalance, they are equal and opp. 19. When on side if canted to right, cr. of gr. is to left of pt. of contact, and tendency is to resume old posn. 20. Each particle is equivalent to two particles of half its wt. at ends of side, and their cr. of gr. is that of triangle. 21. Dis. from each $= \frac{1}{18}(b^2 + c^2 - \frac{1}{2}a^2)$, &c. 22. $\frac{a}{6}\sqrt{19}, \frac{a}{6}\sqrt{13}, \frac{a}{6}\sqrt{7}$. 23. If cr. of gr. of B and D be at E, that of A and C must be there. If neither that of B and D nor of A and C be at E, that of the four cannot be there. 24. Each particle of the second set may be broken into three equal ones in posns. of the particles at whose cr. of gr. it is. 25. For it is cr. of gr. of each set of four. 26. Draw two perp. lines through cr. of gr., and drop perps. on these from the particles. These perps. rept. components of forces, and alg. sum is zero. 27. If O be any pt., G cr. of gr., A posn. of a particle (w), $\Sigma(w.\,OA^2) = OG^2\,\Sigma(w) + \Sigma(w.\,GA^2)$, and $\Sigma(w.\,OA^2)$ is lst. when $OG = 0$. 29. If $AD = \frac{1}{3}AC$, result. of OA, OC is $2\,OD$, and then result. of OB and $2\,OD$ must trisect BD in G say, and $= 3\,OG$. 30. In case of a parallelogram. 31. Indirect proof. 33. Result. of forces passes through cr. of gr., *i.e.*, forces are equal. 36. $\frac{1}{4}\sqrt{73}$, $\frac{3}{4}\sqrt{13}$, $\frac{5}{4}$ 37. Let 2α = vert. angle, θ = angle made by line with base, p = perp. from vertex on base; then line $= \frac{1}{2}p \sin\alpha \left\{\frac{1}{\cos(\theta - \alpha)} + \frac{1}{\cos(\theta + \alpha)}\right\}$. 38. By dividing it into two sets of triangles we may find two lines on each of which it lies, and $\therefore$ on their intersn. 39. $2\,BC$. 40. The intersn. of diags. is cr. of gr., and diags. bisect one another; $\therefore$ fig. is a parallelogram. 41. Sim. to Prob. 38. 42. Let K be the bisection of BD. If $EL = \frac{1}{3}EF$, cr. of gr. is on a line through L parallel to BD; if $KM = \frac{1}{3}KE$, it is on a line through M parallel to AC; and these lines intersect in KF. 46. The cr. of gr. of rods AB, BC, CA is cr. of circle in triangle formed by joining their bisections. Similarly all the way up; $\therefore$ cr. of gr. is in line joining cr. of this circle to the vertex, and is one-third the way up, since that of each triangular side is so. 47. Mts. abt. vertex. On line from vertex to bisection of base, and dist. from vertex $\frac{2}{3}$ of this line. 48. $BD = 2\,PD$. 49. $\frac{1}{3}$ rad. from centre. 50. Dist. from opp. corner $\frac{19}{21\sqrt{2}} \times$ side of sq. 52. Alt. of triangle $= \frac{3 - \sqrt{3}}{2} \times$ side of sq. 53. Compare the sectn. with a triangle perp. to edges, and passing through pt. where sectn. cuts orifice. 54. ABC triangle, O intersn. of perps.; AO may be shewn = twice perp. from cr. of circle on BC. 55. $CA = 2a$, $BC = 2b$; dis $= \frac{1}{a+b}\sqrt{a^4 + b^4 + 2\,a^2\,b^2 \cos ACB}$. 57. Cr.

of gr. of perimeter is that of wts. at D, E, F proportional to the sides, and bisector of FDE divides FE in ratio of AC to AB. 59. D, E, F bisectns. of sides. $PD = \frac{1}{2} AO$, &c. Result. of OA, OB, OC = result. of OD, OE, OF = result. of $\frac{1}{2} AO$ with $OP + \frac{1}{2} BO$ with $OP + \frac{1}{2} CO$ with OP. 60. ABC triangle; D its centre. Take $CE = \frac{1}{3} CB$, and $DG = \frac{1}{7} DE$. G is cr. of gr. 61. Break wt. a into $\frac{1}{2} a$ at B, and $\frac{1}{2} a$ at C, &c. 62. Mts. abt. A. Press. passes through pt. where dirn. of W cuts BC. 64. Mts. abt. vert. pl. through C. 66. Mts. abt. vert. pl. through pt. of suspension. 67. The three forces pass through same pt., and may easily be shewn that angle made by string with reaction of wall = 60°, *i.e.*, by string with vert. = 30°. 70. Shew $2 \sin^2 C = \frac{9a^2 - l_1^2}{4a^2}$, $2 \sin^2 B = \frac{9a^2 - l_2^2}{4a^2}$.

71. α = angle made by strings with one another; β = angle made by string carrying wt. with rod; W = wt. of rod; w = wt. to string. Then $2a \sin(\alpha+\beta) = b \sin \alpha$; $w \sin \beta (a-c) + W \sin (2a+\beta)$. $c + 2w \cos \alpha \sin(\alpha+\beta)(a+c) = o$, $2a$ being length of rod, and c dis. from fixed pt. to mid. of rod. 73. Describe a triangle ABC whose sides are prop. to the wts., and make at cr. angles $B'OC'$, $C'OA'$, $A'OB'$ equal to $180-A$, $180-B$, $180-C$. A', B', C' are pts. reqd. 74. Shew $p = \frac{a}{6} \sin A \cos(B-C)$, &c.

CHAPTER V.—*Levers.*—1. $1\frac{5}{7}$ ft. from 3. 2. P, Q, R pass through a pt., and P, Q are perp. 3. w = suspended wt., W = wt. of CD, G its cr. of gr. $w = W.\frac{CG}{AC} \tan \alpha$. 4. $\frac{a(W+2Q) - b(P+Q+W-R)}{R}$ from P, where R = press. on peg, W = wt. of beam, $2a$ = length of beam, b = dis. of fulcrum from P. 7. Arms are equal. Press. = $4\sqrt{3}$. 8. Incln. of AC to horiz. $= 2\cos^{-1} \frac{P + \sqrt{P^2 + 2(2W+W')^2}}{2(2W+W')}$.

9. As $1 : \sqrt{2}$; 120°. 11. a, b the arms; α, β inclns. to horiz.; P, Q forces; ϕ, ψ their inclns. to horiz.; $a P \sin(\phi - \alpha) = b Q \sin(\psi - \beta)$. 12. 64 cts. 12. $18\frac{3}{7}$, 18 oz. 14. Neither gains nor loses. 16. $\sqrt{W_1 W_2}$; $\frac{1}{2}(W_1 + W_2)$. 19. Loss per cent. = $\frac{100(a-b)^2}{a^2+b^2}$. 20. P must be moved further out, *i.e.*, customer is defrauded. 21. The pt. moves nearer fulcrum; moves further from fulcrum. 22. Lb. graduations are at intervals of $\frac{3}{4}$ in. 23. $Q.AC = P.CN + W.CG$; $W.CG = P.CD$. 24. The graduations will indicate n times the wts. they did before.

Pulleys.—1. R = rad. of wheel, r = rad. of axle: line joining pt. of attachment to cr. makes angle $\sin^{-1} \frac{r}{R}$ with vertical. 2. Increased. 3. 1 lb. Strings make equal angles with vertical. Resolve tensions vertically and horizontally. 4. $1 : 2 \sin \alpha$. 5. $\frac{1}{3}$ his wt. 6. 100 lbs. 7. $\frac{3}{5}$ his wt. if he pull upwards; $\frac{7}{5}$ his wt. if downwards. 8. The cord from each pulley bisects angle bet. cords supporting it. 9. Start at bottom. Let $W = P$; tension of each string must = P. 10. $3\frac{1}{3}$ lbs. 11. $3\frac{1}{2}$ oz. 12. 1 lb.

14. $\frac{7}{8}$ lb. supports pulleys; $6\frac{1}{8}$ lbs. the wt. 15. § 67, Cor. 2. 16. 756 lbs. 18. 1 : 247. 19. $3\frac{1}{4}$ lbs. 20. § 68, Cor., Cor. 1.

Inclined Plane.—1. 2 lbs. 2. 4. 3. 15 lbs. 4. 140. 5. $60\sqrt{3}$. 6. 9 lbs. 7. 14, 50 lbs. 8. $\sqrt{3}$ lbs., 30°. 9. 39, 65 lbs. 10. $3\frac{3}{4}$ lbs. 11. 60°. 13. $W\dfrac{\cos(\theta+\alpha)}{\cos\theta}$; $\sin^{-1}\left(\dfrac{P}{W}\cos\theta\right)$. 14. $\mathrm{Cos}^{-1}\left(\dfrac{W}{P}\sin\alpha\right)$. 18. $\mathrm{Cos}^{-1}\frac{5}{7}$, $\sin^{-1}\frac{1}{3}$. 19. $2\tan^{-1}\dfrac{P}{3W-P}$. 21. $\mathrm{Sin}^{-1}\dfrac{PR}{Wr}$. 22. If P be first power, $4P\cos 60°$ is second and most advantageous. 23. 25 lbs. 24. As $\tan\alpha' : \tan\alpha$, α, α' being inclns. of pls. 25. $\dfrac{w}{n}$ has greatest effect when $\theta = 0$. 26. Let G be cr. of gr.; GE perp. to BC: $BE = \frac{1}{3}(a + c\cos B)$, $GE = \frac{1}{3}c\sin B$, and just as it topples $GBE = 90° - \alpha$. 27. Take mts. abt. pt. abt. which pl. turns. 28. W, W' wts.; α, α' inclns. of pls; w wt. of string, a its length : length of string on pl. of incln. $\alpha = a\dfrac{(W'+w)\sin\alpha' - W\sin\alpha}{w(\sin\alpha+\sin\alpha')}$. 29. W = wt. of beam, θ = angle it makes with pl., ϕ = angle string makes with vertical: $W = P.\dfrac{\sin(\phi+\alpha)}{\sin\alpha}$, giving ϕ; and taking moments abt. pt. where beam meets pl., $\tan\theta = \dfrac{2(P^2 - W^2\sin^2\alpha)^{\frac{1}{2}} - W\cos\alpha}{W\sin\alpha}$. 30. w = wt. of cyl.: revolving vert. and horiz. $w + 2W\cos^2\frac{1}{2}\theta = R\cos\alpha$, $2W\cos\frac{1}{2}\theta\sin\frac{1}{2}\theta = R\sin\alpha$, R being reaction of pl. Eliminate R.

Screw.—1. 119 in. nearly. 2. 24 : 1. 3. 1·07 lbs. nearly. 4. ·366 lbs nearly. 5. $\tan^{-1}\frac{2}{15}$. 6. 48. 7. 16·54 in. nearly. 8. $2\pi mn$. 9. 45°. 10. $n\sqrt{3}$. 11. In each case = $\cot\alpha\,.\,\dfrac{\text{rad. of pr.}}{\text{rad. of cyl.}}$.

Virtual Velocities.—1. 30 in. $W = 5P$, and $W \times$ its rise $= P \times$ its descent. 2. 1 ft. 4. 6. 5. Let a displacement (h) be made along the plane. Now $P\cos\alpha = W\sin\alpha$; $\therefore Ph\cos\alpha = Wh\sin\alpha$; $\therefore P \times$ displacement in dirn. of $P = W \times$ displacement in dirn. of W. Simly. for 2nd part. 6. 2nd part. Suppose a displacement (h) along pl.; then by v. v., $P \times h\cos\theta = W \times h\sin\alpha$, or $P\cos\theta = W\sin\alpha$. 7. Let A be old posn., B new posn. of point, and P, Q, R make angles α, β, γ with AB. Then $p = AB\cos\alpha$, &c., and eq. holds if $P.AB\cos\alpha + Q.AB\cos\beta = R.AB\cos\gamma$; if $P\cos\alpha + Q\cos\beta = R\cos\gamma$; § 23. 8. Until diam. is incd. three times. Previous wt. is to wt. now supported as sum of radii to dif. of radii. 9. p = displt. of P; $\therefore$ that of W in dirn. of $W = p.\dfrac{r}{R}\sin\alpha$, $\therefore p.P = p.\dfrac{r}{R}\sin\alpha.W$. 10. $\dfrac{1}{2\sqrt{3}}$ ton. 11. Make displt. by letting one string remain stationary, and other move parallel to itself. 12. Let angle made by P with string change from α to $\alpha - k$, where k is so small that $\sin k = k$, $\cos k = 1$ nearly. $W = \frac{1}{2}\tan\alpha\,P$.

Work.—1. $\frac{1}{60}$. 2. $9\frac{1}{11}$; with friction $9\frac{1}{11} + \frac{30\sqrt{3}}{11}$. 3. The work is that of raising $1\frac{999}{1000}\pi$ lbs. through $240 + 241\frac{1}{3} + 242\frac{2}{3} + \ldots\ldots 478\frac{2}{3}$ inches. Ans. 3.20 ... min.

Friction.—(Note: the coeff. of fric. is usually denoted by μ). 1. $\frac{1}{\sqrt{3}}$, § 96. 2. Fric. $= 15 \times \sin$ (incl. of pl.) $= 9$. 3. Fric. $= \mu W$, and $W = \mu W$; $\therefore \mu = 1$, and angle reqd. is 45°. 4. $\mu = \tan 45° = 1$; $R = \frac{10}{\sqrt{2}} = 5\sqrt{2}$, and fric. $= \mu R = 5\sqrt{2}$. 5. From second statement $\sin \alpha = \frac{3}{5}$, $\cos \alpha = \frac{4}{5}$; and $R = 10 \times \frac{4}{5} = 8$, $3 + \mu R = 10 \times \frac{3}{5} = 6$, $\therefore$ press. $= \sqrt{R^2 + \mu^2 R^2} = \sqrt{73}$. 6. $R = \frac{W \cos(\alpha + \theta)}{\cos\theta - \mu \sin\theta}$ when body is just sliding down, and $= \frac{W \cos(\alpha + \theta)}{\cos\theta + \mu \sin\theta}$ when body is just sliding up, and when motion is about to take place μ is greatest. 7. $\mu = \frac{1}{\sqrt{3}}$; $R = W \cos 30° = W.\frac{\sqrt{3}}{2}$; fric. $= \mu R = \frac{1}{2} W$; force reqd. $= W \cos 60° + \mu R = \frac{1}{2} W + \frac{1}{2} W = W$. 8. *i.e.* find μ, $= \frac{1 + \sqrt{3} - \sqrt{2}}{\sqrt{3} - 1}$, and $\alpha = \tan^{-1} \frac{1 + \sqrt{3} - \sqrt{2}}{\sqrt{3} - 1}$. 9. $\frac{95}{12}$ ft. Take vert. and horiz. components, and mts. abt. an end. 10. a, b distances from lower and upper end of cr. of gr.; incln. $= \tan^{-1} \frac{a - b\mu\mu'}{\mu(a + b)}$. 11. $\frac{1}{2}$. 12. $\frac{1}{3}$. 13. $\tan\alpha = \mu = \frac{W \sin\theta - P\cos\theta}{W\cos\theta + P\sin\theta}$; $\theta = \alpha + \tan^{-1}\frac{P}{W}$. 14. $\tan^{-1}\left(\frac{1}{\mu} + 2\tan\alpha\right)$, where α is the inclination of the rod. 15. $3000\sqrt{3}$; $\frac{\sqrt{3}}{4}$; $10\sqrt{3}$. 16. Pl. smooth, reaction $= W\frac{\cos(\theta + \alpha)}{\cos\theta}$; pl. rough, $= W\frac{\cos(\theta + \alpha)}{\cos\theta} \cdot \frac{1}{1 + \mu\tan\theta}$, or $= W\frac{\cos(\theta + \alpha)}{\cos\theta} \cdot \frac{1}{1 - \mu\tan\theta}$. 17. $\mu_1 + \mu_2\sqrt{3} = \sqrt{3} - 1$. 18. $\mu_1 + 2\mu_2 = 1$. 20. $90 - \alpha$. 22. Without fric. $T = \frac{W}{2\sqrt{3}}$; with fric., $= \frac{W}{2}\left(\frac{1}{\sqrt{3}} - \frac{1}{2}\tan 15°\right)$. Thrust of leg $= \frac{W}{\sqrt{3}}$. 23. $\cos(\theta + \epsilon) = \frac{W}{P}\cos\epsilon$, where θ is incln. of string to vertical. 24. $\theta =$ angle bet. rod and line of greatest slope, $\alpha =$ incln. of ring to horiz., then $\sin\theta = \mu\cot\alpha$. 26. μW and W, where W is wt. of cyl. 29. $\frac{c\tan^{-1}\mu}{a - c\tan^{-1}\mu} W$.

DYNAMICS.

CHAPTER I.—1. 5 mls. per hr. east. 2. 38·8 .... 3. $\frac{1}{8}$. 4. $\frac{9}{11}$ second. 5. $1\frac{1}{2}$ second; $85\frac{1}{2}$. 6. $\frac{60a}{n}$ ft. 7. 2·19 ft. 8. 38400. 9. $\frac{(60)^4}{5280}f$. 10. If x be the acceln. when the units in wh. s, t, σ, τ are expressed are the units of space and time; then $f = \frac{t^2 x}{s}$, $F = \frac{\tau^2 x}{\sigma}$. 12. 32. 13. 64. 14. Part. comes to rest in 10 units of time, space descrd. being 500. It retraces its path, and in 10 units of time is in its first posn. Afterwards vel. and space descrd. increase indefinitely. 15. 400. 16. $\frac{a+b}{2(b-a)}$. 17. $v = \sqrt{2fs}$, and $\therefore$ equal vels. are *not* generated in equal spaces. 18. s represents dis. from initial pt., and after time $\frac{u}{f}$, body is retracing its path. 19. § 18. 21. $\frac{P-Q}{p-q}$; $\frac{Q(2p-1)-P(2q-1)}{2(p-q)}$. 22. $\frac{1}{2}(v-u)t$. § 9. 24. 16"; 1. 25. 10; 5 ft. 26. $b^2-a^2 : v^2-u^2$. 27. $\frac{v^2-u^2}{2f}$. 28. 5. 29. 56·85 ft. 30. Whole space — $\frac{1}{2}f + (f+\frac{1}{2}f) + (f+\frac{1}{2}f+\frac{1}{4}f) + (f+\frac{1}{2}f+\frac{1}{4}f+\frac{1}{8}f) + \ldots$; and this series may be summed by taking last terms of each bracket; next second to last, &c.

Parallelogram of Velocities.—1. sin$^{-1}\frac{1}{4}$ with 4; 5. 3. v, and makes angle 120° with orig. dirn. 4. Chords are proportional to the vels., and dr. through their intersn. represents resultant vel. 5. 1467·95.... 6. ·23 ml. below oppo. pt. 7. 53° 8′. 8. Bring one body to rest, impressing on both a vel. equal and opp. to it. 9. A, B, the vessels; bring A to rest by impressing vel. V on both; BD the present dirn. of B's motion, C the intersn. of the dirns. of motion at first; then $\frac{\sin\theta}{\sin DBA} = \frac{(V^2+V'^2-2VV'\cos\alpha)^{\frac{1}{2}}}{v}$, and DBA is known, BAC being so.

CHAPTER II.—*Mass.*—1. $30 = mg$, $P = m \times 50 = 46{\cdot}58$ lbs. 2. $\frac{a}{g}\cdot n$. 4. Consider force of gravity. The moving force is expressed by product of number of units of mass in body, and number of units of acceln. it can produce in body. Now, unit of fce. being 1 lb., a fce. of W lbs. is expressed by W; acceln. by 32·2; $\therefore$ $W = m \times 32{\cdot}2$; *i.e.*, m is the number of times 32·2 is contained in W; $\therefore$ unit of mass is mass of 32·2 lbs. 5. Mass of $53\frac{3}{4}$ lbs. 6. $\frac{1}{4}$ second, if $g = 32$. 7. s ft., t units of space and time; $s = 32{\cdot}2t^2$. 8. P lbs. produce an acceln. in 1" of fg feet per second on a mass

equal to that of $m \times 62{\cdot}5$ lbs. But P produces on a mass equal to that of P lbs. an acceln. g; $\therefore$ P produces on $m \times 62{\cdot}5$ an acceln. $\frac{P}{m \times 62{\cdot}5}g$, and this $= fg$; $\therefore$ $P = 62{\cdot}5\,mf$. 9. $2{\cdot}5''$ nearly.

10. $\frac{1}{2} \cdot \frac{4}{5}\,g \cdot t^2 = 49$; vel. $= \frac{4}{5}\,g \cdot t$. 11. 4 : 3. 12. $\frac{1}{2} \cdot \frac{n}{m}\,g \cdot t^2$; $\frac{n}{m}\,g \cdot t$. 13. 1 : 2; 2 : 1. 14. $\frac{33}{160}g$. 15. $8 - \frac{96}{g} = 5$ lbs. nearly. 16. $n\left(1 + \frac{f}{g}\right)$. 17. $\frac{ag}{62{\cdot}5n}$.

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1. (1) By 1st law motn. is uniform and in a st. line. (2) By 2nd law, since moving force varies and mass is const., acceln. varies. (3) Acceln. same for all (2nd law). 2. By 2nd law.

Free Rectilineal Motion.—1. Constant. § 8, Cor. 2; § 9, Cor. 2.

2. $v_1^2 = 2fs_1$, $v_2^2 = 2fs_2$; $\therefore$ $\frac{v_2 + v_1}{2}(v_2 - v_1) = f(s_2 - s_1) = fh$.

3. $\frac{g}{16}$ ft. 5. $t = \sqrt{\frac{2a}{g'}} - \sqrt{\frac{2a}{g}}$, $m = \sqrt{-2ga} - \sqrt{2g'a}$. 6. Vel. $= 40$; accel. $= 20$ and 72000. 7. $3''$. 8. $32 : 32{\cdot}2$; $21\frac{9}{11}$. 9. $32{\cdot}2$; $5''$. 10. $26\frac{2}{3}\frac{5}{6}$ ($g = 32$). 11. $8''$; $161m$. 12. $82\frac{2}{15}$ ft. 13. $\frac{9}{16}$ length of line from top. 14. 1 : 2. 16. $\frac{150}{g}$ lbs. 17. $3622{\cdot}5$ ft.

18. Time through m^{th}. space $= \sqrt{\frac{2m}{g}} - \sqrt{\frac{2(m-1)}{g}}$.

Motion on Inclined Planes.—1. $\sqrt{4g}$. 2. $\sin^{-1}\frac{(\text{vel.})^2}{2g \times \text{length}}$.

3. $t^2 = \frac{2h}{g \sin^2 a}$. 4. $l\sqrt{2}$; $\sqrt{\frac{8l\sqrt{2}}{g}}$; $\sqrt{\frac{2(a + l\sqrt{2})}{g}} + \sqrt{\frac{2l\sqrt{2}}{g}}$ seconds. Gravity would vary sensibly for dif. heights.

5. $\sin^{-1}(\sqrt{2} - 1.)$ 6. $\frac{100\sqrt{10}}{g} + \frac{25g}{2\sqrt{10}} - 50$. 7. Length of string $= \frac{u^2}{4g \sin a}$; time of ascent $= \frac{u}{g \sin a}\left(1 - \frac{1}{\sqrt{2}}\right)$; of descent $= \frac{u}{g \sin a} \cdot \frac{1}{\sqrt{2}}$; reqd. time $= \frac{u}{g \sin a}$. 8. On perp. on base as dr. describe a circle: pts. where it cuts sides are pts. reqd. 9. $v^2 = 2g \cdot \frac{(chd.)^2}{dr.}$. 11. (i.) Describe a circle touching given line and a horizontal line through given pt. at given pt.; chd. joining pts. of contact is reqd. line. (ii.) Describe a circle touching given circle and a horizontal line through given pt. at given pt.

(Todhunter's Euclid, Appendix, § 9). (iii.) Similar to (ii). (iv.) Wherever the pt. be in the line, the line of quickest descent must pass through highest pt. of circle. Describe a circle touching the given line and the given circle at its highest pt. Line through pt. of contact with given line and highest pt. of given circle is reqd. line. Make use of the following problem. 12. $PQ = (AC - AB)$ cos QAC; accel. down $PQ = g$ cos QAC. 13. $\sqrt{3} - 1 : 2$. 14. $t^2 = \frac{(dr.)^3}{2g} \cdot \frac{1}{AC^2}$. 15. If D be dr. of circle which touches horizontal line through S, and which the given circle touches internally, d that of a similar circle which touches the given circle externally, sums of sqs. of times $= \frac{2}{g}(D + d)$. 16. (sin $\alpha - \mu$ cos α) g. 17. Accel. down pl. $= g$ (sin 30° + tan 15° cos 30°) $= \frac{2g}{\sqrt{3}+1}$. Space described $= g(\sqrt{3} + 1)$. 18. 9 ft. 19. Incln. of $AP = \alpha$, of $AQ = \beta$; accel. down $AP = \frac{g \sin \alpha}{\cos(\alpha - \beta)}$. Time down $AP = \sqrt{\left\{2\ AP \cdot \frac{\cos(\alpha - \beta)}{g \sin \beta}\right\}}$; down $AQ = \sqrt{\frac{2\ AQ}{g \sin \beta}}$; which are equal if AP cos $(\alpha - \beta) = AQ$, *i.e.* $AQP = 90°$. 20. Retardation going up $= \frac{g \sin(\alpha + \epsilon)}{\cos \epsilon}$; accel. coming down $= \frac{g \sin(\alpha - \epsilon)}{\cos \epsilon}$.

Projectiles.—1. $9g$. 2. 45°. Resistance of air. 3. Cos 2 α : 1 − sin 2 α. 5. Vert. and and horiz. distances are ·6 and ·8 ft. 6. It has moved vertically downwards $12\frac{1}{2}\ g$ ft., and horizontally 60 g ft.; vel. $= 13\ g$; dirn. of motn. makes angle $\tan^{-1} \frac{5}{12}$ with horizon. 7. 45°. 8. AB, BC the horiz. and vert. vels.; draw AD making given angle with AC, cutting BC produced if necessary in D : BD is reqd. vert. vel. 9. Pts. where it would occur are below horizon. 10. $\frac{v^2 - v'^2}{2g}$. 11. Horiz. dis. bet. them at time $t = (v' - v)$ cos $\alpha \cdot t$; vert. dis. $= (v' - v)$ sin $\alpha \cdot t$; $\therefore$ tan (incln. of line joining them to horiz.) $=$ tan α. 12. Time of B's coming over $A = \frac{AB}{v \cos \alpha}$; then A is at ht. $= u \cdot \frac{AB}{v \cos \alpha} - \frac{1}{2} g \left(\frac{AB}{v \cos \alpha}\right)^2$; and B is at ht. $= v \sin \alpha \cdot \frac{AB}{v \cos \alpha} - \frac{1}{2} g \left(\frac{AB}{v \cos \alpha}\right)^2$: and these must be equal; $\therefore \sin \alpha = \frac{u}{v}$. 13. $\frac{1}{2} g t^2 - v \sin \alpha \cdot t$, where $t = \frac{2v}{g} \cdot \frac{\cos \beta}{\sin(\alpha + \beta)}$. 14. $g = 32$. 16. Dis. from $A = \frac{16 v^2}{25 g}$. 19. 45°; $\tan^{-1} \frac{1}{\sqrt{2}}$. 23. tan (incln. of line joining them

to horiz.) $= \dfrac{v' \sin \alpha' - v \sin \alpha}{v \cos \alpha - v' \cos \alpha'}$, and is const. 24. (vel.)$^2 = 4\,g$ × rad. 25. Is equal to $\dfrac{v \cos \alpha}{g}(\tan \theta - \tan \phi)$. If t be time to P, T time in arcs, tangents of inclinations are $\dfrac{v \sin \alpha - g\,t}{v \cos \alpha}$, $\dfrac{v \sin \alpha - g\,(t+T)}{v \cos \alpha}$, $\dfrac{v \sin \alpha . g\,(t+2\,T)}{v \cos \alpha}$, &c. 27. A highest pt., B, C pts. of section of tangent with horizontal and vertical lines, M pt. in horizon vertically below A; $\dfrac{v^2 \sin^2 \alpha}{g} - C\,M = v \sin \alpha \cdot t - \frac{1}{2}\,g\,t^2$, and C is uniformly acceld.; $B\,A = \dfrac{v \cos \alpha}{2\,g}(v \sin \alpha - gt)$; and B moves uniformly. 29. $\dfrac{2\,v^2}{g} \tan 10° \sec 10°$. 30. In both cases $= \dfrac{2\,v^2 \cos \alpha \sin (\alpha - \theta)}{g \cos^2 \theta}$. 31. Let $\alpha =$ inclination of plane to horizon, $\beta =$ angle dirn. of projection makes with plane; then range $= \dfrac{v^2}{g \cos^2 \alpha}\{\sin (\alpha + 2\,\beta) - \sin \alpha\}$, wh. is gst. when $\alpha + 2\,\beta = 90$, or $\beta = \frac{1}{2}(90 - \alpha)$. 33. $v\,t$; $v\sqrt{\dfrac{2\,s}{g \sin \alpha}}$. 34. Let t be time of reaching deck; then range on deck $= v \cos \alpha . t$, t being given by $-h = v \sin \alpha \cdot t - \frac{1}{2}\,g\,t^2$. 37. True dist. is $a\,\dfrac{\sin 2\,(\alpha+\theta) - \sin 2\,\alpha}{2 \cos^2 (\alpha + \theta)}$ $= a\,\dfrac{\cos 2\,\alpha}{\cos^2 \alpha} \cdot \theta$, approx. 40. Dis. $= \dfrac{v^2}{2g}$.

Falling Bodies Connected by a String.—1. $6 + \dfrac{12}{7g}$, $6 - \dfrac{12}{7g}$; for $P + Q = 12$, and $\dfrac{P-Q}{P+Q}\,g = f = \frac{2}{7}$. 2. $3\frac{1}{3}$ or 30. 3. $g\,(2\,n - 1) + 2a : g\,(2n - 1) - 2a$. 4. 30 in. per second; 8″; 192 ft. 5. 18″. After taking up 7 lbs., system comes to rest in $6\frac{3}{4}$″ and then moves back. It would return to the pt. in 8″. 7. $\dfrac{P^2 - Q^2}{Q}$. 8. If u be the vel. of each, spaces are $\dfrac{u^2}{2g}$, $\dfrac{3\,u^2}{2\,g}$. 9. $\dfrac{P}{P+Q}\,g$; $\left(\dfrac{P}{P+Q}\right)^2 \cdot g$, $\dfrac{PQ}{(P+Q)^2} \cdot g$. 10. If T be the tension $\dfrac{Pg - T}{P} =$ accel. of $P =$ accel. of $Q = \dfrac{T - Qg}{Q}$; $\therefore\ T = \dfrac{2\,PQ}{P+Q} \cdot g$. 13. Keep $P + Q$ constant and vary $P - Q$, and observe distances for same times. $\frac{1}{3}\frac{2}{4}\frac{5}{4}\,g$; $2\frac{1}{17}$ lbs. 14. $4\,P =$ mass of string. When part is cut off $\dfrac{2\,Pg - T}{2\,P} =$ accel. of longer part $=$ that of shorter $= \dfrac{T - Pg}{P}$;

$\therefore T = \frac{4Pg}{3}$ and pressure on peg $= \frac{8Pg}{3}$. 15. $Qg \cos \alpha \cdot \frac{P - Q \sin \alpha}{P+Q}$, α being inclin. of pl. 16. The vels. are $\frac{1}{2n+1} yt$, $\frac{2^2}{2n+1} yt$, &c. 18. Vert. accel. of cr. of gr. $= \frac{1}{4} g (\sin \alpha - \sin \beta)^2$, horiz. accel. $= \frac{1}{4} g (\sin \alpha - \sin \beta) (\cos \alpha + \cos \beta)$. 19. $(\text{Vel.})^2 = g c \cdot \frac{r(r-1)}{n}$. When the particles become a continuous string $(\text{vel.})^2$ when nth leaves $= g \cdot \frac{a}{n-1} \cdot \frac{n(n-1)}{n} = a g$. 20. $a =$ ht. of wedge, $b =$ its base, $v =$ vel. of ball, $V =$ vel. of wedge. Then $mv = MV$; vert. vel. on reaching ground $= \sqrt{2ag}$; $\therefore v = \sqrt{\frac{ag}{2}}$; time to ground $= \sqrt{\frac{2a}{g}}$. Horiz. distance descrd. by ball $= \sqrt{\frac{2a}{g}} \cdot \sqrt{\frac{ag}{2}} = a$. $\therefore V \sqrt{\frac{2a}{g}} + a = b$, $\frac{m}{M} \sqrt{\frac{ag}{2}} \cdot \sqrt{\frac{2a}{g}} + a = b$, &c.

Impact.—1. Common vel. after impact $= \frac{1}{7}$; momenta, $1\frac{2}{7}$, $1\frac{5}{7}$. 3. $\frac{A}{B}$. 4. $\frac{v(1+e)}{1-e}$, opp. to gn. vel. v. 5. If the balls be of equal mass, and elasticity perfect. 6. 30°. 7. $Au + Bv = (A+B) w = Au' + Bv'$, to eliminate A and B. 8. $u = \frac{Ax + Bv}{A+B}$. $u = \frac{Ax - Cv}{A+C}$, to eliminate A and x. 9. Communicated to the earth. They rise $e^2 s$, s being ht. descended. 10. $\frac{2a}{u-v}$, a being dis. bet. gn. pts., and u, v gn. vels. 11. Ball continues to rebound for time $\sqrt{\frac{2a}{g}} (1 + 2e + 2e^2 + \ldots$ ad. inf.$) = 3 \sqrt{\frac{2a}{g}}$ 12. $6g$, $6g\sqrt{2}$, and $6(2 - \sqrt{2})$ seconds. 14. Distances are a, $2ae^2$, $2ae^4$, &c.; times $\sqrt{\frac{2a}{g}}$, $2e\sqrt{\frac{2a}{g}}$, $2e^2\sqrt{\frac{2a}{g}}$, &c. 15. Ranges are $\frac{v^2}{g} \sin 2\theta$, $\frac{v^2}{g} \sin 2\theta \cdot e$, $\frac{v^2}{g} \sin 2\theta \cdot e^2$, &c. 17. tan (angle of incidence) $= \frac{3e - e^3}{\sqrt{3}(1+e)}$. 18. First angle of incidence $= \tan^{-1} \sqrt{\frac{e^3}{1+e+e^2}}$. 22. Collns. occur at distces. $\frac{u}{u+v} \times$ Oce from one another, and balls will come tog. at A when mult. of $\frac{u}{u+v} \times$ Oce

= mult. of ◯ce. 25. $\tan\theta_{n+1} = \frac{1}{e^n}\tan\theta_1$; $v_{n+1} = v_1 \cdot e^n$. $\frac{\cos\theta_1}{\cos\theta_{n+1}} = v_1 \cdot e^n \cdot \frac{1}{e^n} \cdot \frac{\sin\theta_1}{\sin\theta_{n+1}} = v_1 \sin\theta_1$, ultly. 29. Vel. of $A = \frac{u(A - Be)}{A + B}$; vel. of $B = u\sqrt{1 + \left\{\frac{A(1+e)}{A+B}\right\}^2}$; B makes angle $\tan^{-1}\frac{A+B}{A(1+e)}$ with dirn. of A's motion. 30. A moves off perp. to line joining crs. with vel. $u\frac{\sqrt{3}}{2}$; B moves along this line with vel. $\frac{u}{2}$.

THE END.

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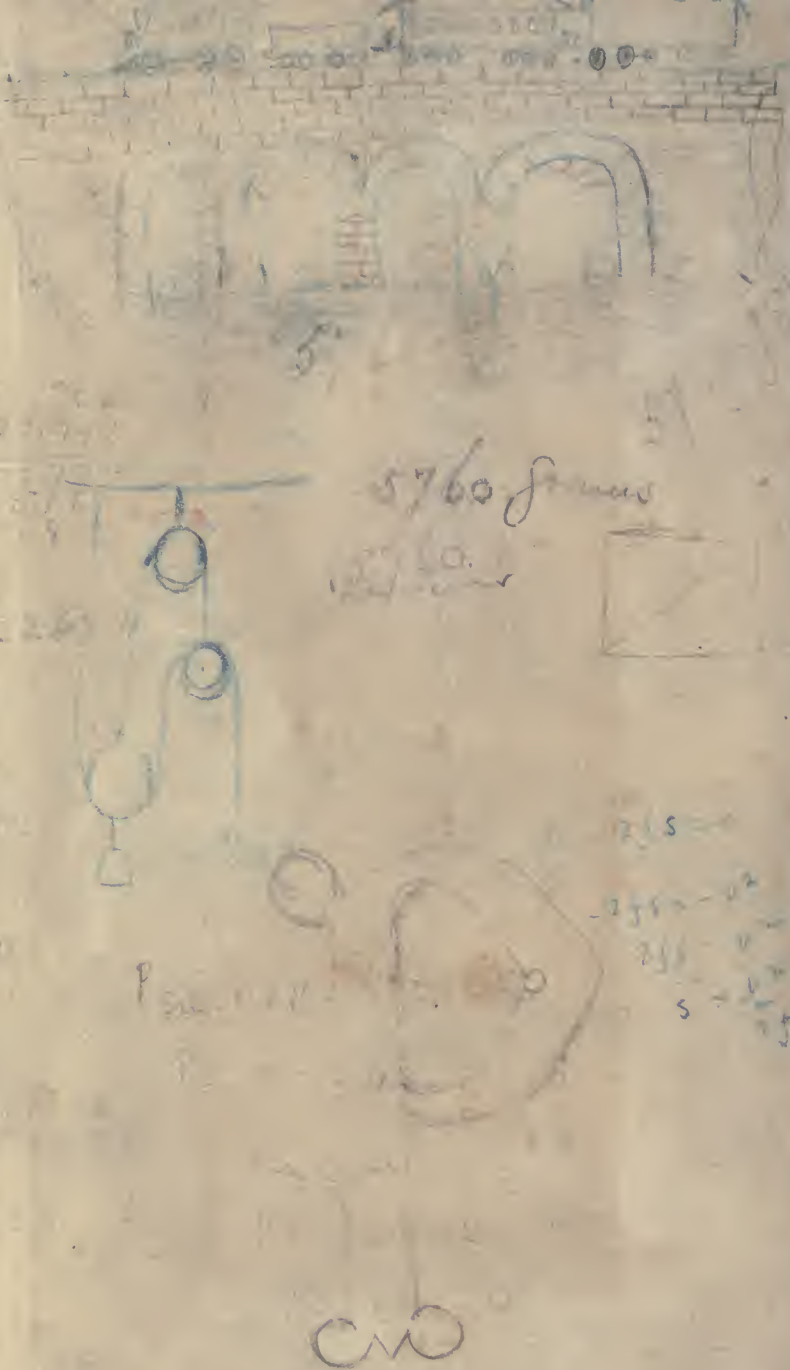

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